

# The Ising model and Special Geometries

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**Abstract.** We show that the globally nilpotent  $G$ -operators corresponding to the factors of the linear differential operators annihilating the multifold integrals  $\chi^{(n)}$  of the magnetic susceptibility of the Ising model ( $n \leq 6$ ) are homomorphic to their adjoint. This property of being self-adjoint up to operator homomorphisms, is equivalent to the fact that their symmetric square, or their exterior square, have rational solutions. The differential Galois groups are in the special orthogonal, or symplectic, groups. This self-adjoint (up to operator equivalence) property means that the factor operators we already know to be Derived from Geometry, are special globally nilpotent operators: they correspond to “Special Geometries”.

Beyond the small order factor operators (occurring in the linear differential operators associated with  $\chi^{(5)}$  and  $\chi^{(6)}$ ), and, in particular, those associated with modular forms, we focus on the quite large order-twelve and order-23 operators. We show that the order-twelve operator has an exterior square which annihilates a rational solution. Then, its differential Galois group is in the symplectic group  $Sp(12, \mathbb{C})$ . The order-23 operator is shown to factorize in an order-two operator and an order-21 operator. The symmetric square of this order-21 operator has a rational solution. Its differential Galois group is, thus, in the orthogonal group  $SO(21, \mathbb{C})$ .

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## 1. Introduction

In previous papers [1, 2, 3, 4] some massive calculations have been performed on the magnetic susceptibility of the square Ising model, in particular, on the  $n$ -particle (multifold integrals) contributions  $\tilde{\chi}^{(n)}$  of the susceptibility. For instance, the linear differential operator for  $\tilde{\chi}^{(5)}$ , was carefully analyzed [2, 3], and similar calculations were achieved [4] on  $\chi^{(6)}$ .

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Among the various factor operators in the factorization of the minimal order linear differential operators for  $\tilde{\chi}^{(5)}$  and  $\tilde{\chi}^{(6)}$ , some are found to be associated with elliptic functions, others are found to have a *modular form interpretation* [5, 6], and an order-four operator, emerging for  $\chi^{(6)}$ , is found to be associated to a Calabi-Yau ODE (associated with a  ${}_4F_3$  hypergeometric function with an *algebraic pull-back*). There remain two linear differential operators of large orders (twelve and twenty-three) which are too involved to seek for a possible modular form, or Calabi-Yau interpretation.

If the occurrence of linear differential operators *associated with elliptic curves* for square Ising correlation functions, or *modular forms*, is not a surprise [7], the kind of linear differential operators that should emerge in quite involved *highly composite* objects, like the  $n$ -particle components  $\chi^{(n)}$  of the susceptibility of the square Ising model, is far from clear. We just had a prejudice, inherited from the Yang-Baxter integrability of the Ising model, that they should be "special" and could possibly be associated with elliptic curves<sup>‡</sup>. The result [6] on an order-four operator of  $\chi^{(6)}$  which is a Calabi-Yau ODE, clearly shows that one moves away from the elliptic curve framework, and that the Ising model *does not restrict to the theory of elliptic curves* [9] (and their associated elliptic functions and *modular forms*).

The integrand of the multifold integrals of Ising model is *algebraic* in the variables of integration and in the other remaining variables. As a consequence we know that these multifold integrals can be interpreted as "*Periods*" of *algebraic varieties* and verify *globally nilpotent* [5] linear differential equations<sup>†</sup>: they are [10, 11, 12] "*Derived From Geometry*". In a recent paper [13] we also show that the multifold integrals of Ising model *actually correspond to diagonals of rational functions*, and this remarkable property does explain, may be not the modularity property [6] of these  $n$ -fold integrals, but, at least, some *integrality* (resp. globally bounded) property of the corresponding series [13].

Inside this "Geometry" framework [14], the multifold integrals of Ising model seem to be even more "selected": this justifies to explore these "Special Geometries".

Actually, in a previous paper [15, 16], and with a learn-by-example approach, we displayed a set of enumerative combinatorics examples corresponding to miscellaneous *lattice Green functions* [17, 18, 19, 20, 21, 22, 23], as well as Calabi-Yau examples, together with order-seven operators [24, 25] associated with differential Galois groups which are exceptional groups. On the *irreducible* operators of these examples, two differential algebra properties occur simultaneously [15]. On the one hand, these operators are *homomorphic to their adjoint*, and, on the other hand, their symmetric, or exterior, squares have a *rational solution* [15]. These properties are equivalent, and correspond to special differential Galois groups. The differential Galois groups are not the  $SL(N, \mathbb{C})$ , or extensions of  $SL(N, \mathbb{C})$ , groups one could expect generically, but *selected*  $SO(N, \mathbb{C})$ ,  $Sp(N, \mathbb{C})$ ,  $G_2$ , ... differential Galois groups [26].

An irreducible linear differential operator  $L_q$ , of order  $q$ , has, generically, a symmetric square ( $Sym^2(L_q)$ ) of order  $N_s = q(q+1)/2$  and an exterior square ( $Ext^2(L_q)$ ) of order  $N_e = q(q-1)/2$ . If the Wronskian of  $L_q$  is rational and  $Sym^2(L_q)$  annihilates a rational solution, or is of order  $N_s - 1$ , the group is in the *orthogonal group*  $SO(q, \mathbb{C})$  that admits an invariant quadratic form. If the Wronskian of  $L_q$  is rational and  $Ext^2(L_q)$  has a rational solution, or is of order  $N_e - 1$ , the group is in the *symplectic group*  $Sp(q, \mathbb{C})$  that admits an invariant alternating form, and the

<sup>‡</sup> Corresponding to the canonical parametrization of the Ising model [8] in terms of elliptic functions

<sup>†</sup> These linear differential operators factorize into irreducible operators that are also necessarily globally nilpotent [5].

order  $q$  is necessarily *even*.

We are going to use these tools on the globally nilpotent operators of the  $n$ -particle (multifold integrals) contributions of the magnetic susceptibility of the Ising model, and show that these operators are not only “Derived from Geometry”, but actually correspond to “Special Geometries”.

The paper is organized as follows. In Section 2, we recall the factorizations of the linear differential operators corresponding to the linear differential equations of  $\tilde{\chi}^{(5)}$  and  $\tilde{\chi}^{(6)}$ . In Section 3, we show (and recall) that all of the “small order” factor operators are *associated with elliptic functions*, have a modular form interpretation, or are Calabi-Yau ODE’s. These factor operators are, therefore, homomorphic to their adjoints. Sections 4 and 5 are devoted to the “large order” linear differential operators which occur for  $\tilde{\chi}^{(5)}$  and  $\tilde{\chi}^{(6)}$ . Seeking homomorphisms of these “large” factors with their corresponding adjoints is out of our current computer resources. Instead, keeping in mind the results of [15], we look for, and produce, the rational solutions of their exterior and symmetric squares. Section 6 displays a set of (quite technical) remarks on the subtleties of these massive calculations. Section 7 contains the conclusion.

## 2. Recalls

### 2.1. The linear differential equation of $\tilde{\chi}^{(5)}$

With series of 10000 terms (modulo a prime), we have obtained [1] the Fuchsian differential equation annihilating  $\tilde{\chi}^{(5)}$ , which is of order 33. Subsequently [2], it was shown that the linear combination of 5, 3 and 1-particle contributions to the magnetic susceptibility

$$\Phi^{(5)} = \tilde{\chi}^{(5)} - \frac{1}{2}\tilde{\chi}^{(3)} + \frac{1}{120}\tilde{\chi}^{(1)}, \quad (1)$$

is annihilated by an order twenty-nine linear ODE. The corresponding linear differential operator  $L_{29}$ , factorizes as

$$L_{29} = L_5 \cdot L_{12}^{(\text{left})} \cdot \tilde{L}_1 \cdot L_{11}, \quad (2)$$

with

$$L_{11} = (Z_2 \cdot N_1) \oplus V_2 \oplus (F_3 \cdot F_2 \cdot L_1^s). \quad (3)$$

The linear differential equations have been obtained in primes, and we have obtained in exact arithmetic some factors occurring in the factorization. All the factors have been reconstructed and are known *in exact arithmetic*, except of  $L_{12}^{(\text{left})}$ ,  $L_5$  and  $\tilde{L}_1$ , which are known *only modulo some primes*. The linear differential operator  $L_5$  is irreducible. Its analytical solution (at 0) has been written [1] as a homogeneous polynomial of (homogeneous) degree 4 of the complete elliptic integrals  $K$  and  $E$ .

Then, considering the inhomogeneous equation

$$L_{24}(\Phi^{(5)}) = \text{sol}(L_5), \quad (4)$$

where

$$L_{24} = L_{12}^{(\text{left})} \cdot \tilde{L}_1 \cdot L_{11}, \quad (5)$$

we have reconstructed [3], in exact arithmetic,  $L_{24}$  and  $L_5$ , and have shown that  $L_{12}^{(\text{left})}$  is *irreducible* [3].

### 2.2. The linear differential equation of $\tilde{\chi}^{(6)}$

The order-52 linear differential equation of  $\tilde{\chi}^{(6)}$ , and the factorization of the corresponding linear differential operator, have been given in [4]. It was shown that the linear combination of 6, 4 and 2-particle contributions to the magnetic susceptibility

$$\Phi^{(6)} = \tilde{\chi}^{(6)} - \frac{2}{3}\tilde{\chi}^{(4)} + \frac{2}{45}\tilde{\chi}^{(2)}, \quad (6)$$

is annihilated by an order forty-six linear ODE. The corresponding linear differential operator  $L_{46}$ , factorizes as

$$L_{46} = L_6 \cdot L_{23} \cdot L_{17}, \quad (7)$$

with

$$L_{17} = L_4^{(4)} \oplus \left( D_x - \frac{1}{x} \right) \oplus L_3 \oplus (L_4 \cdot \tilde{L}_3 \cdot L_2), \quad (8)$$

$$L_4^{(4)} = L_{1,3} \cdot (L_{1,2} \oplus L_{1,1} \oplus D_x),$$

where  $D_x$  denotes the derivative with respect to  $x$ .

The linear differential equation has been obtained in primes, and we have obtained, in exact arithmetic, some factor operators occurring in the factorization. All the factors are known *in exact arithmetic*, except of  $L_{23}$  and  $L_6$  which are known *only modulo some primes*. While  $L_6$  is irreducible, since its analytical solution (at 0) has been written as a polynomial expression of homogeneous degree 5 of the complete elliptic integrals  $K$  and  $E$ , we have not reached [4] any conclusion on whether the operator  $L_{23}$  is reducible or not. Performing the factorization based on the combination method presented in Section 4 of [2], needs prohibitive computational times.

### 2.3. Sum up

We have a plenty<sup>†</sup> of linear differential operators occurring as factor operators in the linear differential equations of  $\tilde{\chi}^{(5)}$  and  $\tilde{\chi}^{(6)}$ . Beyond the order-one differential operators,  $L_1$ ,  $N_1$  (given in [27]),  $L_1^s$ ,  $\tilde{L}_1$  (given in [2, 3]), and the fully factorizable order-four operator  $L_4^{(4)}$  (given in [4] and its solution in [28]), there are linear differential operators of higher orders  $Z_2$ ,  $V_2$ ,  $F_2$ ,  $F_3$ ,  $L_5$  and  $L_{12}^{(\text{left})}$  for  $\tilde{\chi}^{(5)}$ , and  $L_2$ ,  $L_3$ ,  $\tilde{L}_3$ ,  $L_4$ ,  $L_6$  and  $L_{23}$  for  $\tilde{\chi}^{(6)}$ .

The next section deals with these small order operators (up to order 6) where we show (and/or recall) for each one, that it is either equivalent to a symmetric power of  $L_E$ , the differential operator corresponding to the complete elliptic integral  $E$ , or has a symmetric (or exterior) square which annihilates a rational solution. Next, it is shown that each operator  $L_q$  is *homomorphic to its adjoint* [15]

$$L_q \cdot R_n = \text{adjoint}(R_n) \cdot \text{adjoint}(L_q), \quad (9)$$

and we focus on the order of  $R_n$ , and on the coefficient in front of the higher derivative. Note that, for irreducible operators, there is another equivalence relation

$$\text{adjoint}(L_q) \cdot S_p = \text{adjoint}(S_p) \cdot L_q, \quad (10)$$

<sup>†</sup> Typically these problems can be rephrased in terms of variation of mixed Hodge structures. To some extent, this explains the fact that the minimal order operators annihilating the  $\tilde{\chi}^{(n)}$ 's factorize in a *quite large number of factor operators*.

which sends the solutions of  $L_q$  into the solutions of the adjoint. In the sequel, we will consider the relation (9).

**Notation:** The linear differential operators of  $\tilde{\chi}^{(5)}$  and  $\tilde{\chi}^{(6)}$  have large orders and factorize in many factors. With a few exception, all the factor operators carry large degree polynomials whose roots are *apparent singularities*. In the sequel we adopt the notation  $A_n(L_q)$  for these apparent polynomials of a linear differential operator  $L_q$ ,  $n$  denoting the degree of the “apparent” polynomial.

### 3. Rational solution for $Sym^2$ or $Ext^2$ versus Homomorphism with the adjoint

#### 3.1. Rational solution for $Sym^2$ or $Ext^2$

The order-two operator  $V_2$  (given in [2]) is equivalent to the second order operator associated with  $\tilde{\chi}^{(2)}$  (or equivalently to  $L_E$  the linear differential operator corresponding to the complete elliptic integral of the second kind  $E$ ). Similarly, the order-two operator  $L_2$  (given in [4]) is equivalent to the second order operator associated with  $\tilde{\chi}^{(2)}$ . This is also the case for the order-three operator  $L_3$  (given in [4]) which is equivalent to the symmetric square of the second order operator associated with  $\tilde{\chi}^{(2)}$ . The equivalence occurs also for the order-five operator  $L_5$ , occurring [2] in  $\tilde{\chi}^{(5)}$ , which is the *symmetric fourth power* of  $L_E$ , and the order-six operator  $L_6$ , occurring [4] in  $\tilde{\chi}^{(6)}$ , which is the *symmetric fifth power* of  $L_E$ . There are also the order-three operator  $Y_3$  (given in [29]), and the order-four operator [5, 28]  $M_2$ , which are symmetric second (resp. third) power of  $L_E$ .

For all the linear differential operators which are symmetric powers of  $L_E$ , their solutions are given as polynomials of homogeneous degree in the complete elliptic integrals. For those which are equivalent to  $L_E$  (or to the symmetric power of  $L_E$ ), the solutions can be written as homogeneous polynomials in the complete elliptic integrals and their derivatives [2, 4].

Other operators are equivalent to hypergeometric functions *up to rational pullbacks*. The order-two operator  $Z_2$  (given in [30]), occurring (also) in the factorization of the linear differential operator [27] associated with  $\tilde{\chi}^{(3)}$ , is seen to correspond to a *modular form of weight one* [5]. The order-two operator  $F_2$  (given in [2]) corresponds to a *modular form*: its solutions can be written in terms of Gauss hypergeometric functions with pullbacks [6].

Now there are linear differential operators of order  $\geq 3$  which are equivalent to symmetric powers of hypergeometric functions with a pullback.

##### 3.1.1. Symmetric square of $F_3$

The symmetric square of the order-three operator  $F_3$  is an order-six linear differential operator which is a direct sum of an order-five and an order-one differential operators. The rational solution of the symmetric square of  $F_3$ , denoted  $S_R(Sym^2(F_3))$ , reads

$$S_R(Sym^2(F_3)) = \frac{P_{34}(x)}{D_{26}(x) \cdot A_7(F_2)^2}, \quad (11)$$

with

$$D_{26}(x) = x^2 \cdot (x-1)^2 (1+2x)^2 (4x-1)^9 (1+4x)^7 (4x^2+3x+1)^2,$$

where  $P_{34}(x)$  is a polynomial of degree 34 given in Appendix A, and where the degree-seven apparent polynomial for the order-two operator  $F_2$  appearing in (3), reads:

$$A_7(F_2) = 1 + x - 24x^2 - 145x^3 - 192x^4 + 96x^5 + 128x^7. \quad (12)$$

The irreducible linear differential operator  $F_3$  has a symmetric square of order six that annihilates a rational function. The differential Galois group of the operator  $F_3$  is in the orthogonal group  $SO(3, \mathbb{C})$ , or equivalently the group  $PSL(2, \mathbb{C})$ . The differential operator  $F_3$  is equivalent (up to a multiplicative function) to the symmetric square of an order-two differential operator<sup>§</sup>. The independent solutions of this order-two operator (call it  $O_2$ ) are

$$x^{-1/3} \cdot {}_2F_1\left(\left[\frac{1}{6}, \frac{1}{6}\right], \left[\frac{1}{2}\right]; P_1\right), \quad x^{1/3} \cdot \sqrt{P_1} \cdot {}_2F_1\left(\left[\frac{2}{3}, \frac{2}{3}\right], \left[\frac{3}{2}\right]; P_1\right), \quad (13)$$

with:

$$P_1(x) = \frac{1}{108} \frac{(1-4x^2)(1+32x^2)^2}{x^2} = \frac{1}{108} \frac{(1-16x^2)^3}{x^2} + 1. \quad (14)$$

The solutions of  $F_3$  can be written in terms of the solutions (13) of this order-two operator  $O_2$  (the order-two intertwiner  $R_2$  is given in Appendix B)

$$\text{sol}(F_3) = (1-4x)^{-9/2} (1+4x)^{-7/2} \cdot R_2\left(\text{sol}(\text{Sym}^2(O_2))\right). \quad (15)$$

Anticipating some comments in section 6, we give the rational solution of the *symmetric square of the adjoint* of<sup>†</sup>  $F_3$

$$S_R(\text{Sym}^2(\text{adjoint}(F_3))) = \frac{N_{33}(x) \cdot P_{53}(x)}{A_{37}(F_3)^2}, \quad (16)$$

with:

$$N_{33}(x) = x^5 \cdot (1-x)^2 (1-2x)^2 (1+2x)^4 (1-4x)^{10} \times (1+4x)^6 (1+3x+4x^2)^2. \quad (17)$$

The degree-37 apparent polynomial  $A_{37}(F_3)$  and the degree-53 polynomial  $P_{53}(x)$  are given, respectively, in (A.1) and (A.3).

### 3.1.2. Symmetric square of $\tilde{L}_3$

Similarly, the order-three differential operator  $\tilde{L}_3$  has a symmetric square differential operator of order-six which is a direct sum of an order-five and an order-one linear differential operators, the latter annihilating the rational solution  $S_R(\text{Sym}^2(\tilde{L}_3))$ :

$$S_R(\text{Sym}^2(\tilde{L}_3)) = \frac{2 - 42x + 225x^2 - 260x^3 - 4352x^4 + 49152x^5}{(1-16x)^7}. \quad (18)$$

The linear differential operator  $\tilde{L}_3$  is equivalent (up to a multiplicative function) to the symmetric square of an order-two differential operator. The independent solutions of this order-two operator (call it  $O_2$ ) are

$${}_2F_1\left(\left[\frac{1}{8}, \frac{3}{8}\right], \left[\frac{1}{2}\right]; P_1\right), \quad \sqrt{P_1} \cdot {}_2F_1\left(\left[\frac{5}{8}, \frac{7}{8}\right], \left[\frac{3}{2}\right]; P_1\right), \quad (19)$$

<sup>§</sup> See [31] for the reduction of order-three ODE to order-two ODE.

<sup>†</sup> Here  $F_3$  is taken to be monic: its head coefficient of  $D_x^3$  is normalized to 1.

with

$$P_1(x) = \frac{(1-12x)^2}{(1-16x)(1-4x)^2} = \frac{256x^3}{(1-16x)(1-4x)^2} + 1. \quad (20)$$

The solutions of  $\tilde{L}_3$  can be written in terms of the solutions (19) of the order-two operator (the order-two intertwiner  $R_2$  is given in Appendix B):

$$\text{sol}(\tilde{L}_3) = \frac{(1-16x)^{9/2}(1-4x)^{3/2}}{x^2 \cdot (1024x^3 - 1232x^2 + 160x - 5)} \cdot R_2\left(\text{sol}(\text{Sym}^2(O_2))\right). \quad (21)$$

The rational solution of the *symmetric square of the adjoint* of  $\tilde{L}_3$  (taken in monic form) reads:

$$S_R(\text{Sym}^2(\text{adjoint}(\tilde{L}_3))) = \frac{x^4 \cdot (1-16x)^6 \cdot P_{10}(x)}{(1-4x)^3 \cdot A_4(\tilde{L}_3)^2}, \quad (22)$$

where the degree-four apparent polynomial of  $\tilde{L}_3$ ,  $A_4(\tilde{L}_3)$  reads:

$$A_4(\tilde{L}_3) = 4352x^4 + 3607x^3 - 1678x^2 + 252x - 8. \quad (23)$$

The polynomial  $P_{10}$  is given in (A.4).

### 3.1.3. Exterior square of $L_4$

The last of the “small order” linear differential operators is the order-four operator  $L_4$  occurring in the linear differential operator of  $\tilde{\chi}^{(6)}$ . The linear differential operator  $L_4$  has been analyzed in [6] and was shown to be equivalent to a *Calabi-Yau equation* with  ${}_4F_3$  hypergeometric function with an *algebraic pullback*. This algebraic pullback is simply related to the modulus of the elliptic functions parametrizing the Ising model. The solution of this order-four operator  $L_4$ , is sketched in Appendix B.3.

The exterior square of  $L_4$  is of order six. It is a *direct sum* of an order-five and an order-one differential operators. The order-one operator annihilates the rational solution,  $S_R(\text{Ext}^2(L_4))$ , that we recall [6]

$$S_R(\text{Ext}^2(L_4)) = \frac{P_{17}(x)}{x^9 \cdot (1-16x)^{13}(1-4x)^2 \cdot A_4(\tilde{L}_3)}, \quad (24)$$

where  $A_4(\tilde{L}_3)$  is given in (23). The degree-17 polynomial  $P_{17}$  is given in (A.6).

The rational solution of the exterior square of the adjoint of  $L_4$  reads:

$$S_R(\text{Ext}^2(\text{adjoint}(L_4))) = \frac{x^{11} \cdot (1-16x)^{14}(1-4x)^2(1-8x) \cdot P_{17}(x)}{A_{26}(L_4)}. \quad (25)$$

The degree-26 apparent polynomial  $A_{26}(L_4)$  is given in (A.5).

### 3.2. Homomorphism with the adjoint

All the previous linear differential operators ( $V_2$ ,  $L_2$ ,  $L_3$ ,  $L_5$ ,  $L_6$ ) which are homomorphic to  $L_E$ , or homomorphic to the symmetric square of  $L_E$ , are naturally homomorphic with their adjoints. This is straight consequence of the homomorphism of  $L_E$  with its adjoint.

The more subtle linear differential operators ( $Z_2$ ,  $F_2$ ,  $F_3$ ,  $\tilde{L}_3$ ), which have been shown [6] to be associated with *modular forms*, and more precisely  ${}_2F_1$  hypergeometric

functions up to rational (or algebraic) pull-backs, are also homomorphic to their adjoint. For instance,  $Z_2$  is conjugated to its adjoint

$$\begin{aligned} Z_2 \cdot W_Z(x) &= W_Z(x) \cdot \text{adjoint}(Z_2), & \text{with:} \\ W_Z(x) &= \frac{(1+2x) \cdot (1-x) \cdot (96x^4 + 104x^3 - 18x^2 - 3x + 1)}{(1+4x)^2 \cdot (1-4x)^5 \cdot (1+3x+4x^2) \cdot x}, \end{aligned} \quad (26)$$

where  $W_Z(x)$  is the Wronskian of  $Z_2$ .

Similarly,  $F_2$  is homomorphic to its adjoint

$$\begin{aligned} F_2 \cdot W_F(x) &= W_F(x) \cdot \text{adjoint}(F_2), & \text{with:} \\ W_F(x) &= \frac{x \cdot A_7(F_2)}{(1+3x+4x^2)^2 \cdot (1+4x)^3 \cdot (1-4x)^6}, \end{aligned} \quad (27)$$

where, again,  $W_F(x)$  is the Wronskian of  $F_2$ , and where  $A_7(F_2)$  is the apparent polynomial of  $F_2$  given in (12).

### 3.3. Homomorphism with the adjoint for $F_3$

The order-three linear differential operator  $F_3$  is homomorphic to its adjoint, with a large order-two intertwiner

$$\begin{aligned} F_3 \cdot R_2 &= \text{adjoint}(R_2) \cdot \text{adjoint}(F_3), & \text{with} \\ R_2 &= a_2(x) \cdot D_x^2 + a_1(x) \cdot D_x + a_0(x), \end{aligned} \quad (28)$$

where  $a_2(x)$  is the *rational solution* of  $\text{Sym}^2(F_3)$  given in (11), and where

$$a_1(x) = \frac{P_{78}(x)}{\rho_1(x) \cdot A_7(F_2)^2 \cdot A_{37}(F_3)}, \quad a_0(x) = \frac{P_{122}(x)}{\rho_1(x) \cdot \rho_0(x) \cdot A_7(F_2)^2 \cdot A_{37}(F_3)^2},$$

with

$$\begin{aligned} \rho_0(x) &= x \cdot (x-1) (16x^2-1) (4x^2-1) (4x^2+3x+1), \\ \rho_1(x) &= x^3 \cdot (x-1)^3 (2x-1) (4x-1)^{10} (1+4x)^8 (1+2x)^3 (4x^2+3x+1)^3, \end{aligned}$$

and where  $A_7(F_2)$  is given in (12) and  $A_{37}(F_3)$  is given in (A.1). The various polynomials  $P_j(x)$  are of degree  $j$ .

### 3.4. Homomorphism with the adjoint for $\tilde{L}_3$

The order-three linear differential operator  $\tilde{L}_3$  is also homomorphic to its adjoint

$$\begin{aligned} \tilde{L}_3 \cdot R_2 &= \text{adjoint}(R_2) \cdot \text{adjoint}(\tilde{L}_3), & \text{with} \\ R_2 &= a_2(x) \cdot D_x^2 + \frac{P_{11}(x)}{(16x-1)^7 \cdot \rho(x) \cdot A_4(\tilde{L}_3)} \cdot D_x + \frac{P_{17}(x)}{(16x-1)^7 \cdot \rho(x)^2 \cdot A_4(\tilde{L}_3)^2}, \end{aligned} \quad (29)$$

where

$$\rho(x) = x \cdot (4x-1) (16x-1), \quad (30)$$

and where  $a_2(x)$  is the *rational solution* of  $\text{Sym}^2(\tilde{L}_3)$  given in (18).



3.5. Homomorphism with the adjoint for  $L_4$ 

Now, let us consider the last “small order” linear differential operator occurring in  $\tilde{\chi}^{(6)}$ , namely the order-four operator  $L_4$ . It is also homomorphic with its adjoint with an *order-two* intertwiner

$$L_4 \cdot R_2 = \text{adjoint}(R_2) \cdot \text{adjoint}(L_4), \quad \text{with} \quad (31)$$

$$\begin{aligned} R_2 = & a_2(x) \cdot D_x^2 + \frac{P_{46}(x)}{\rho_1(x) \cdot A_4(\tilde{L}_3) \cdot A_{26}(L_4)} \cdot D_x \\ & + \frac{P_{75}(x)}{\rho_1(x) \cdot \rho_0(x) \cdot A_4(\tilde{L}_3) \cdot A_{26}(L_4)^2}, \end{aligned} \quad (32)$$

where

$$\rho_0(x) = x \cdot (4x-1)(8x-1)(16x-1), \quad (33)$$

$$\rho_1(x) = x^{10} \cdot (4x-1)^3(8x-1)(16x-1)^{14}, \quad (34)$$

where  $A_4(\tilde{L}_3)$  is the degree-four apparent polynomial of  $\tilde{L}_3$  given in (23),  $A_{26}(L_4)$  of degree 26 is the apparent polynomial of  $L_4$  given in (A.5), and where  $a_2(x)$  is the *rational solution* of  $\text{Ext}^2(L_4)$  given in (24). The polynomials  $P_j(x)$  are of degree  $j$ .

Note that  $R_2$ , the order-two intertwiner (31), is almost self-adjoint: it is such that

$$Y_2^{(L)} = R_2 \cdot \frac{1}{r(x)} = \alpha_2(x) \cdot D_x^2 + \frac{d\alpha_2(x)}{dx} \cdot D_x + \alpha_0(x), \quad (35)$$

is an order-two *self-adjoint* operator, where  $r(x)$  is the rational function:

$$r(x) = 1080 \cdot \frac{P_{26} \cdot A_4(\tilde{L}_3)}{P_{17}^2 \cdot (1-8x)(1-16x) \cdot x^2}. \quad (36)$$

Since  $Y_2^{(L)}$  is self-adjoint, the Wronskian of  $Y_2^{(L)}$  is also equal to  $1/\alpha_2(x)$ , the inverse of the head coefficient of  $Y_2^{(L)}$ . The Wronskian of  $Y_2^{(L)}$  is equal to the rational function  $r(x)^2 \cdot S_R(\text{Ext}^2(\text{adjoint}(L_4)))$ , where  $S_R(\text{Ext}^2(\text{adjoint}(L_4)))$  is (25), the rational solution of the adjoint of  $L_4$  (written in monic form).

The other homomorphisms of  $L_4$  with its adjoint corresponds to the intertwining relation

$$\text{adjoint}(L_2) \cdot L_4 = \text{adjoint}(L_4) \cdot L_2, \quad (37)$$

where, again, the order-two operator  $L_2$  is almost self-adjoint: it is such that

$$Y_2^{(R)} = r(x) \cdot L_2 = \beta_2(x) \cdot D_x^2 + \frac{d\beta_2(x)}{dx} \cdot D_x + \beta_0(x), \quad (38)$$

is *self-adjoint*, where  $r(x)$  is the *same* rational function (36). The Wronskian of  $Y_2^{(R)}$  is nothing but  $S_R(\text{Ext}^2(L_4))$  given by (24). Since  $Y_2^{(R)}$  is self-adjoint, the Wronskian of  $Y_2^{(R)}$  is also equal to  $1/\beta_2(x)$ , the inverse of the head coefficient of  $Y_2^{(R)}$ .

**Remark:** Recalling the miscellaneous decompositions  $L_n \cdot L_m + \text{Cst}$  (up to an overall function), obtained for the lattice Green operators displayed in [15, 16], and since the order-four operator  $L_4$  is homomorphic to its adjoint with order-two intertwiners (see (31), (37)), it is tempting to find such a  $L_n \cdot L_m + \text{Cst}$  decomposition

for  $L_4$ . One easily deduces [15, 16] the following decomposition for  $L_4$ , written in monic form:

$$L_4 = r(x) \cdot (Y_2^{(L)} \cdot Y_2^{(R)} + 1), \quad (39)$$

where  $Y_2^{(L)}$  and  $Y_2^{(R)}$  are the two (quite large) *self-adjoint operators* (35) and (38), and where  $r(x)$  is the previous rational function (36).

Since  $L_4$  in (39) is monic, one has the following relation between the head coefficients of the two self-adjoint operators  $Y_2^{(L)}$  and  $Y_2^{(R)}$ :

$$r(x) \cdot \alpha_2(x) \cdot \beta_2(x) = 1 \quad \text{or:} \quad \frac{1}{\beta_2(x)} = r(x) \cdot \alpha_2(x) = a_2(x), \quad (40)$$

where  $a_2(x)$  is the head coefficient of  $R_2$  in (31).

One easily verifies that  $S_R(\text{Ext}^2(L_4)) = 1/\beta_2(x)$  (see (24)) is solution of the exterior square of the order-two operator  $Y_2^{(R)}$ . The rational function  $1/\alpha_2(x)$  is solution of the exterior square of the order-two operator  $Y_2^{(L)}$ , we denote  $S_R(Y_2^{(L)})$ , and is simply related to  $S_R(\text{Ext}^2(\text{adjoint}(L_4)))$  (see (25)): it is nothing but  $r(x)^2 \cdot S_R(\text{Ext}^2(\text{adjoint}(L_4)))$ . Actually the exterior square of the adjoint monic order-four operator  $\text{adjoint}(L_4) = (Y_2^{(R)} \cdot Y_2^{(L)} + 1) \cdot r(x)$  has the same rational solution as the exterior square of  $Y_2^{(L)} \cdot r(x)$ , namely  $1/r(x)^2 \cdot S_R(Y_2^{(L)})$ .

These two results are a simple consequence of the fact that the rational solution of the exterior square of a decomposition like (39), is, in general, the rational solution of the right-most order-two operator in the decomposition (see [15, 16]).

### 3.6. Comments

In all these previous examples on the “small order” linear differential operators occurring in the  $\tilde{\chi}^{(n)}$ , we have, as we showed for other examples in [15], the simultaneous occurrence of two properties: the *homomorphism of the irreducible operator with its adjoint* and the *occurrence of a rational solution in the symmetric square, or exterior, square* of the differential operator. The expressions of the intertwiners given above, clearly show this link. Each time, the linear differential operator has a symmetric square (or exterior square) annihilating a rational solution (see (11), (18), (24)), it is precisely, this rational solution that appears in the coefficient of the higher derivative of the intertwiners (see (28), (29), (31)). For order-two linear differential operators, the rational solution of the exterior square is just the Wronskian.

For these small orders ( $\leq 4$ ) examples, one even sees that, for odd (resp. even) order operators, it is the rational solution of the symmetric (resp. exterior) square which builds the intertwiner.

However, beyond these examples, for an order-four linear differential operator which is homomorphic to its adjoint with an odd order intertwiner, it is the symmetric square which is involved. Appendix C shows the situation with *generic* linear differential operators of order three and four. The intertwiners of the homomorphism have rational coefficients only when the symmetric (or exterior) square have rational solutions. When there is homomorphism with the adjoint, for order-three linear differential operators, the intertwiner can be of order two, or a function, and it is the symmetric square which annihilates the rational solution (see (C.4)). For order-four operators, if the intertwiner is of order two, or a function, it is the *exterior square* which annihilates a rational solution (see (C.14)), while for order-one and order-three intertwiners, the rational solution is annihilated by the *symmetric square* (see (C.22)).

Note that we have used the results in Appendix C to build the homomorphisms (28) and (31) of  $F_3$  and  $L_4$ , which are the largest of our small orders differential operators.

We have finished with the small order factors occurring in the linear differential operators of  $\tilde{\chi}^{(5)}$  and  $\tilde{\chi}^{(6)}$ . The differential Galois groups of the order-three operators are in the *orthogonal group*  $SO(3, \mathbb{C})$ . The differential Galois group of the order-four operator  $L_4$  is in the *symplectic group*  $Sp(4, \mathbb{C})$ . Appendix D gives examples of invariant forms in both cases.

We turn now to the “large order” linear differential operators  $L_{12}^{(\text{left})}$  and  $L_{23}$ . Here, and as a consequence of the large size of these operators, the approach for finding the intertwiners between  $L_{12}^{(\text{left})}$  (and  $L_{23}$ ) with their corresponding adjoints is hopeless with our current computational resources. In the following two sections, keeping in mind our results on the “small order” operators, we will claim that  $L_{12}^{(\text{left})}$  will be homomorphic to its adjoint if we find a rational solution annihilated by the exterior square (or by the symmetric square). Similarly,  $L_{23}$  will be homomorphic to its adjoint, if a rational solution of the symmetric square is found.

#### 4. On the linear differential operator $L_{12}^{(\text{left})}$ in $\tilde{\chi}^{(5)}$

To see whether the exterior square of  $L_{12}^{(\text{left})}$  has a rational solution it is simpler to start from the definition of the exterior square. The formal solutions (at 0) of  $L_{12}^{(\text{left})}$  are obtained (modulo a prime) and are, either analytic, or logarithmic with a maximum cubic power  $\ln(x)^3$ . The general solution (at 0) of  $L_{12}^{(\text{left})}$  is written as

$$F_3(x) \cdot \ln(x)^3 + F_2(x) \cdot \ln(x)^2 + F_1(x) \cdot \ln(x) + F_0(x). \quad (41)$$

Some 120 starting terms are needed to generate the series  $F_3(x)$  with an homogeneous recurrence and the other series  $F_j(x)$  with inhomogeneous recurrences. The twelve solutions  $S_p$  are taken as in [3]. We form the linear combination

$$\sum_{k,p} d_{k,p} \cdot \left( S_k \frac{dS_p}{dx} - S_p \frac{dS_k}{dx} \right), \quad k \neq p = 1, \dots, 12, \quad (42)$$

which is a general solution of the order<sup>†</sup> 66 exterior square  $\text{Ext}^2(L_{12}^{(\text{left})})$ . Demanding that this combination should not contain log's, fixes some of the coefficients  $d_{k,p}$ .

For a rational solution of  $\text{Ext}^2(L_{12}^{(\text{left})})$  to exist, the form, which is, now, analytic at  $x = 0$

$$D(x) \cdot \sum_{k,p} d_{k,p} \cdot \left( S_k \frac{dS_p}{dx} - S_p \frac{dS_k}{dx} \right), \quad (43)$$

should be a polynomial, and where  $D(x)$  is the polynomial whose roots are the regular singularities of  $L_{12}^{(\text{left})}$ . Each regular singularity in  $D(x)$  is taken with the power  $n_j$ , ( $n_j$  being twice the maximum local exponent of that singularity in  $L_{12}^{(\text{left})}$ ).

<sup>†</sup> By cancelling all the coefficients in front of each  $x^n \ln(x)^p$  in (42), we find that all the  $d_{k,p}$  are zero, this means that  $\text{Ext}^2(L_{12}^{(\text{left})})$  is of order 66. Similarly, if we carry the same calculations for  $\text{adjoint}(L_{12}^{(\text{left})})$  and find that all the  $d_{k,p}$  in the corresponding combination are zero, this will mean that  $\text{Ext}^2(\text{adjoint}(L_{12}^{(\text{left})}))$  has the order 66. This calculation for  $\text{adjoint}(L_{12}^{(\text{left})})$  has not been done.

Our series  $S_j$  are of length 600 and the coefficients depend on some remaining  $d_{k,p}$ . By canceling the coefficients of the higher terms in (43), all the coefficients down to a given term are automatically zero, and we obtain a polynomial. The rational solution of the exterior square  $Ext^2(L_{12}^{(\text{left})})$  thus reads

$$S_R(Ext^2(L_{12}^{(\text{left})})) = \frac{P_{312}(x)}{A_{131}(\tilde{L}_1 \cdot L_{11}) \cdot D_{211}(x)}, \quad (44)$$

with

$$\begin{aligned} D_{211}(x) = & x^{18} \cdot (2x-1)^2 (x-1)^{12} (x+1)^2 (2x+1)^{13} (4x+1)^{22} (4x-1)^{24} \\ & (4x^2-2x-1)^2 (4x^2+3x+1)^{14} (x^2-3x+1)^2 (8x^2+4x+1)^8 \\ & (4x^3-3x^2-x+1)^6 (4x^3-5x^2+7x-1)^8 (4x^4+15x^3+20x^2+8x+1)^6, \end{aligned}$$

where  $P_{312}(x)$  is a polynomial of degree 312, and where  $A_{131}(\tilde{L}_1 \cdot L_{11})$  is the apparent polynomial of the product  $\tilde{L}_1 \cdot L_{11}$ .

The linear differential operator  $L_{12}^{(\text{left})}$  is *irreducible* [3]. Its exterior square annihilates a rational function. Its differential Galois group is in *symplectic group*  $Sp(12, \mathbb{C})$ . Finding the rational solution of the *exterior square of the adjoint* of  $L_{12}^{(\text{left})}$  is, for the moment, beyond our computer facilities. Recalling, for the order- $q$  lattice Green operators displayed in [15, 16], the decompositions of the type  $L_n \cdot L_m + Cst$  (up to an overall function), where  $L_m$  and  $L_n$  are self-adjoint operators, and where  $m$  and  $n$  are two integers of the same parity, it is tempting to imagine, for  $L_{12}^{(\text{left})}$ , a decomposition of the form  $L_{2n} \cdot L_{2m} + Cst$ , where the exterior square of  $L_{2m}$  would have (44) as a rational solution. This would imply the existence of a rational solution for the *exterior square of the adjoint* of  $L_{12}^{(\text{left})}$ , that identifies with the rational solution of the exterior square of  $L_{2n}$ .

## 5. On the linear differential operator $L_{23}$ in $\tilde{\chi}^{(6)}$

To fit our scheme that the linear differential operators, occurring in the Ising model, correspond to "Special Geometry", the linear differential operator  $L_{23}$  should have a rational solution for its symmetric square, which is equivalent to say that  $L_{23}$  is homomorphic to its adjoint.

To see whether the symmetric square<sup>†</sup> of  $L_{23}$  has a rational solution, the general solution of  $Sym^2(L_{23})$  is built from the formal solutions (mod. prime) of  $L_{23}$  as

$$\sum_{k,p} f_{k,p} \cdot S_k S_p, \quad k \geq p = 1, \dots, 23, \quad (45)$$

which should contain neither log's, nor  $x^a$ , ( $a$  half integer), thus fixing some of the coefficients  $f_{k,p}$ .

For a rational solution of  $Sym^2(L_{23})$  to exist, the form (analytic at  $x = 0$ )

$$D(x) \cdot \sum_{k,p} f_{k,p} \cdot S_k S_p, \quad k \geq p = 1, \dots, 23, \quad (46)$$

should be a polynomial, where the denominator  $D(x)$  reads

$$\begin{aligned} D(x) = & x^{n_1} \cdot (1-16x)^{n_2} (1-4x)^{n_3} (1-9x)^{n_4} (1-25x)^{n_5} (1-x)^{n_6} \\ & \times (1-10x+29x^2)^{n_7} (1-x+16x^2)^{n_8}, \end{aligned}$$

<sup>†</sup> By cancelling all the coefficients in front of each  $x^n \ln(x)^p$  and  $x^a$ ,  $a$  half integer in (45), we find that all the  $f_{k,p}$  are zero, which means that  $Sym^2(L_{23})$  is of order 276.

the order of magnitude of the exponents  $n_j$  being obtained from the local exponents of the singularities.

With very long series, we have found no rational solution for  $Sym^2(L_{23})$ . If we trust that our series are long enough, we face one of two situations. Either  $L_{23}$  is irreducible and does not follow the general scheme of being special geometry, contrary to all the other operators (obtained right now) in the Ising model, or the linear differential operator  $L_{23}$  is reducible. In the latter situation (i.e.  $L_{23}$  is *reducible*), the right factor in  $L_{23}$  should be of order even: let us fix this order to two. The linear differential operator  $L_{23}$  is assumed to have the factorization

$$L_{23} = L_{21} \cdot \tilde{L}_2. \quad (47)$$

In this case, and assuming that the factorization (47) is unique,  $Sym^2(L_{23})$  does not need to have a rational solution, but its exterior square *should have* an order-one right factor, since (see Remark 7 below)

$$Ext^2(L_{23}) = O_{252} \cdot Ext^2(\tilde{L}_2). \quad (48)$$

The next step is, then, to see whether it is the exterior square of  $L_{23}$  which has a rational solution. The general solution of  $Ext^2(L_{23})$  is written as

$$\sum_{k,p} d_{k,p} \cdot (S_k \frac{dS_p}{dx} - S_p \frac{dS_k}{dx}), \quad k \neq p = 1, \dots, 23, \quad (49)$$

and should not contain log's and  $x^a$ , ( $a$  half integer), fixing some of the coefficients  $d_{k,p}$ .

For a rational solution of  $Ext^2(L_{23})$  to exist, the form (analytic at  $x = 0$ )

$$D(x) \cdot \sum_{k,p} d_{k,p} \cdot (S_k \frac{dS_p}{dx} - S_p \frac{dS_k}{dx}), \quad (50)$$

should be a polynomial. We, indeed, found a rational solution for  $Ext^2(L_{23})$  which has the form

$$\frac{A_{79}(L_{17}) \cdot P_{93}(x)}{x^4 \cdot (1 - 16x)^6}, \quad (51)$$

where  $P_{93}$  is a degree 93 polynomial, and where  $A_{79}(L_{17})$  is the degree-79 apparent polynomial of  $L_{17}$ .

The existence of a rational solution means that an invariant alternating form is preserved, and this implies that the order is even. Since, this is not the case, the differential operator  $L_{23}$  has a right factor of even order. To establish the factorization (47), we should show that the rational solution (51) is for the exterior square of an *order-two* linear differential operator.

### 5.1. The factorization $L_{23} = L_{21} \cdot \tilde{L}_2$

The rational solution of  $Ext^2(\tilde{L}_2)$  does not determine the linear differential operator  $\tilde{L}_2$  whatever its order is. We show now, how we have obtained the linear differential operator  $\tilde{L}_2$ .

Once the combinations of the  $d_{k,p}$  (in (49)) in front of the log's and  $x^a$ , ( $a$  half integer) have been canceled, the rational solution is looked for. This fixes all the  $d_{k,p}$ . We collect (49) over  $dS_j/dx$ , i.e. we consider the linear combination of series in front of  $dS_j/dx$ . Here  $S_j$  is the solution with the maximum power of  $\ln(x)$ , which is  $\ln(x)^4$  for  $L_{23}$ . We get no combination. There is also no combination when we collect over

$dS_j/dx$ , with  $S_j$  the solution with  $\ln(x)^3$  and  $\ln(x)^2$  terms. When  $S_j$  is the solution with a  $\ln(x)$  term, we obtain six identical combinations, i.e. series. If there is a right factor in  $L_{23}$ , this right factor will be of order  $\geq 2$ . We then search the linear ODE, that annihilates the combination found, and find that its order is, in fact, *two*, this is  $\tilde{L}_2$ .

We have then shown that  $L_{23}$  actually has the factorization (47). Acting by  $\tilde{L}_2$  on the solution of  $L_{23}$  gives a series which is used to obtain  $L_{21}$ .

The linear ODE corresponding to  $\tilde{L}_2$  has the (analytic at  $x = 0$ ) solution

$$\frac{(1-16x) \cdot P_{90}(x) \cdot K(x) + P_{91}(x) \cdot E(x)}{x^{13} \cdot (1-16x)^{15} (1-4x)^2 (1-8x) \cdot A_{79}(L_{17})}, \quad (52)$$

where  $P_{90}$  and  $P_{91}$  are polynomials of degree 90 and 91, and where  $K(x)$  and  $E(x)$  are the complete elliptic integrals of the first and second kinds,  $K(x) = {}_2F_1([1/2, 1/2], [1], 16x)$ ,  $E(x) = {}_2F_1([1/2, -1/2], [1], 16x)$ . The linear differential operator  $\tilde{L}_2$  is equivalent to  $L_E$  the differential operator of the elliptic integral  $E(x)$ . The solution (52) is for  $\tilde{L}_2$  appearing in the factorization  $\tilde{L}_2 \cdot L_{17}$ , where all the linear differential operators are *monic*, and of minimal orders.

The rational solution of  $Ext^2(\tilde{L}_2)$  reads

$$\frac{P_{93}(x)}{x^{24} \cdot (1-16x)^{26} (1-4x)^4 (1-8x)^2 \cdot A_{79}(L_{17})}, \quad (53)$$

which is, as it should, the rational solution (51), divided by the square of the polynomial in front of the higher derivative of  $L_{17}$ . Having obtained the linear differential operator  $\tilde{L}_2$ , we can see that the roots of the polynomial  $P_{93}(x)$  are *apparent singularities* of  $\tilde{L}_2$ ,  $A_{93}(\tilde{L}_2) = P_{93}(x)$ .

We turn now, to the linear differential operator  $L_{21}$ , where the same calculations (as for  $L_{23}$ ) are performed on its *symmetric square*. We find that  $Sym^2(L_{21})$  has a rational solution which reads

$$\begin{aligned} S_R(Sym^2(L_{21})) &= \frac{P_{714}(x)}{D_{529}(x)}, \\ D_{529}(x) &= x^{13} \cdot (1-16x)^{56} (1-4x)^{63} (1-9x)^{47} (1-25x)^{63} (1-x)^{47}, \\ &\quad \times (1-10x+29x^2)^{57} (1-x+16x^2)^{63}, \end{aligned} \quad (54)$$

where  $P_{714}$  is a polynomial of degree 714.

With the assumption that  $L_{21}$  is irreducible (see Remark 8, below), the differential Galois group of  $L_{21}$  is seen to be included in the orthogonal group  $SO(21, \mathbb{C})$ .

## 6. Remarks

In this section we give some technical remarks on the computations displayed in Section 4 and Section 5. The way, the operator  $\tilde{L}_2$  has been obtained, is applied to show that  $L_{21}$  is irreducible. By simple arguments based on the number of logarithmic solutions of maximum degree occurring in  $L_{21}$ , we exclude the possibility that  $L_{21}$  can be a symmetric power of an operator of smaller order.

**Remark 1:** From the factorization (5) we obtained  $L_{12}^{(\text{left})}$  by right division of  $L_{24}$  by the order-twelve operator  $\tilde{L}_1 \cdot L_{11}$  in its non monic form, because this is more tractable. This means that the linear differential operator  $L_{12}^{(\text{left})}$  we are using is in

fact  $L_{12}^{(\text{left})} \cdot P_{12}^{(\text{right})}$ , where  $P_{12}^{(\text{right})}$  is the polynomial in front of the derivative  $D_x^{12}$  of  $\tilde{L}_1 \cdot L_{11}$ . To obtain and give the rational solution (44), we have corrected by dividing by the square of  $P_{12}^{(\text{right})}$ . The rational solution (44) is the solution we would have obtained if we have used the "exterior power" then "ratsols" commands of DEtools in Maple, on the differential operator  $L_{12}^{(\text{left})}$  in the factorization  $L_{12}^{(\text{left})} \cdot \tilde{L}_1 \cdot L_{11}$  with  $\tilde{L}_1 \cdot L_{11}$  in monic form.

**Remark 2:** Another remark is that, we can safely use (and we did) a non-minimal order (in this case order 37) ODE for  $L_{12}^{(\text{left})}$ . The minimal order  $L_{12}^{(\text{left})}$  has at the higher derivative, besides the regular singularities, the polynomials  $A_{131}(\tilde{L}_1 \cdot L_{11})^{11} \cdot A_{828}(L_{12}^{(\text{left})})$ , where  $A_{828}(L_{12}^{(\text{left})})$  is the degree 828 apparent polynomial of  $L_{12}^{(\text{left})}$ . The whole coefficient is of degree 2317, to be compared with the degree of the coefficient in front of the higher derivative  $D_x^{37}$  of the non-minimal ODE which is 160. At the formal solutions generation step, there are only twelve solutions that appear. This is because, at the point  $x = 0$ , all the extra and spurious solutions correspond to critical exponents *that are not, in general, rational numbers*. In the modulo prime calculations, these exponents appear as roots of polynomials of degree 2 and higher. When the spurious exponent appears integer, we should redo the calculations with another prime, or change to another non minimal order equation.

**Remark 3:** One remarks that the rational solution given in (54) has not been corrected (as we did in (44), (53), see Remark 1) by dividing by the square of the coefficient of higher derivative of  $\tilde{L}_2 \cdot L_{17}$ , which reads

$$x^{12} \cdot (1 - 8x)(1 - 4x)^2(1 - 16x)^{11} \cdot A_{79}(L_{17})^2 \cdot A_{93}(\tilde{L}_2). \quad (55)$$

This type of correction is done when, in a given factorization, we deal with non monic factors which is more tractable for large operators. In the case of the rational solution (54), this corresponds to  $L_{21}$  annihilating the series obtained by acting with a *non minimal* order-23  $\tilde{L}_2$  on the solution of  $L_{23}$ . To correct (54), as we did for (44) and (53), we should have obtained  $L_{21}$  by using the  $\tilde{L}_2$  in *minimal order*. In this case, the length of the series to encode  $L_{21}$  is very high. However, the occurrence of a rational solution to the symmetric square of  $L_{21}$  can be seen whether we use a minimal order ODE, or a non minimal order ODE<sup>†</sup> for  $\tilde{L}_2$ . Appendix E shows the results on  $\text{Sym}^2(F_3)$  in the factorization  $F_3 \cdot F_2$ , which occurs in the linear differential operator of  $\tilde{\chi}^{(5)}$  in both ways, i.e. the rational solution of  $\text{Sym}^2(F_3)$ , where  $F_3$  is obtained from the factorization  $F_3 \cdot F_2$  with  $F_2$  of minimal order and with  $F_2$  of non minimal order.

**Remark 4:** The rational solution of the symmetric (or exterior) square of the operator  $L_q$  may carry some or all the regular singularities of  $L_q$ . In the expressions of the rational solutions given in (11), (24), (44) and (53), one remarks that some *apparent polynomials* occur. These polynomials are apparent for the operator which is *at the right* of  $L_q$ . For the operator  $L_q$ , these polynomials are poles (see Appendix B.1 in [2]). For the rational solution given in (18) there is no such apparent polynomial because the factor  $L_2$ , at the right of  $\tilde{L}_3$ , has no apparent singularities. Note that the apparent polynomial of the operator, itself, appears in the denominator of the rational solution of the symmetric (or exterior) square when we deal with the adjoint of the operator. If we call  $\rho_j$  the local exponents at a regular (or apparent) singularity

<sup>†</sup> See Appendix E, and [1], for a deeper understanding of the non minimal order representation of an operator.

of  $L_q$ , at the same point the local exponents of  $\text{adjoint}(L_q)$  are  $-\rho_j + q - 1$  (and  $-\rho_j - q + 1$  for the singularity at infinity). An apparent singularity has the local exponents  $0, 1, \dots, q - 2, q$ , for the adjoint one gets automatically a pole.

**Remark 5:** For all our operators  $L_q$  we have obtained a rational solution for the symmetric (or exterior) square which have the maximum order  $N_s = q(q + 1)/2$  (or  $N_e = q(q - 1)/2$ ). For irreducible operator  $L_q$  with differential Galois group in  $SO(q, \mathbb{C})$  (or  $Sp(q, \mathbb{C})$ ), it may happen that  $\text{Sym}^2(L_q)$  (or  $\text{Ext}^2(L_q)$ ) does not have the generic order but has the order  $N_s - 1$  (or  $N_e - 1$ ). This means that there is no rational solution but there is, instead, a relation between the solutions of  $L_q$  (or the solutions of  $L_q$  and their first derivative). This drop in the order of  $\text{Sym}^2(L_q)$  (or  $\text{Ext}^2(L_q)$ ) will also be seen for  $\text{Sym}^2(\text{adjoint}(L_q))$  (or  $\text{Ext}^2(\text{adjoint}(L_q))$ ) if  $L_q$ , and its adjoint, are conjugated, i.e. are homomorphic with a function. When  $L_q$  and its adjoint are homomorphic with an intertwiner (not a function), if  $\text{Sym}^2(L_q)$  (or  $\text{Ext}^2(L_q)$ ) have the order  $N_s - 1$  (or  $N_e - 1$ ),  $\text{Sym}^2(\text{adjoint}(L_q))$  (or  $\text{Ext}^2(\text{adjoint}(L_q))$ ) will have the order  $N_s$  (or  $N_e$ ) and annihilate a rational solution (details will be given elsewhere).

**Remark 6:** We have succeeded to factorize the linear differential operator  $L_{23}$  via the rational solution of  $\text{Ext}^2(L_{23})$ . This, then, completes the factorization method we forwarded in section 4 of [2]. Recall that, in this method, we produce the general (analytic at 0) solution of  $L_{23}$  which begins as

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots \quad (56)$$

with the higher coefficients depending on the  $a_j$ , ( $j = 0, \dots, 5$ ). We let the coefficients vary in the range  $[1, p_r]$ ,  $p_r$  being the prime, until a differential equation of order less than 23 is found. If this happens, there is a right factor to  $L_{23}$ . This computation should have required the maximum time of  $2p_r^5 t_0$  as necessary to produce  $\tilde{L}_2$ , if  $t_0$  denotes the time needed to obtain  $L_{23}$ . The  $d_{k,p}$  coefficients, we mentioned in paragraph before (52), are precisely the actual values of the  $a_j$  in (56) for which the series (56) will be solution of an order-two ODE.

**Remark 7:** Note that (47) and (48) are an obvious property of the exterior power, which states that if  $L_q = L_{q-n} \cdot L_n$ , then the exterior power  $\text{Ext}^n(L_q)$  will have the order-one right factor

$$\text{Ext}^n(L_n) = D_x - \frac{d}{dx} \ln(W(x)) \quad (57)$$

where  $W(x)$  is the Wronskian of  $L_n$ . For our purposes, it happens that the suspected right factor (i.e.  $\tilde{L}_2$ ) is of *order two*, and we are dealing with the *second* exterior power. If our suspected right factor in  $L_{23}$  were not of order two, we would still use

$$\text{Ext}^2(L_{23-n} \cdot L_n) = O_{253-n(n-1)/2} \cdot \text{Ext}^2(L_n) \quad (58)$$

which is a general identity and we would expect the rational solution of  $\text{Ext}^2(L_{23})$  to come from  $\text{Ext}^2(L_n)$  and use the recipe (paragraph before (52)) to obtain  $L_n$ .

**Remark 8:** The general (analytic at 0) solution of  $L_{21}$  begins as

$$b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + \dots \quad (59)$$

with the higher coefficients depending on the  $b_j$ , ( $j = 1, \dots, 5$ ). We have then four coefficients to vary, which is very time consuming. The way we have factorized  $L_{23}$  can be repeated for  $L_{21}$ . Here also, once the coefficients in  $\sum f_{k,p} S_k S_p$  have been fixed to encode the rational solution (54), we collect over the series  $S_j$ , which is in  $\ln(x)^4$ .



We obtain two series ( $L_{21}$  has two series in  $\ln(x)^4$ , see Remark 9). If there is a right factor to  $L_{21}$ , it should be of order  $\geq 5$ . If it exists, its solution is a combination of both series. This way, we have reduced the ODE search from varying four coefficients to one coefficient. The computation time is still high, but the calculation can be done in parallel on many subintervals of  $[1, p_r]$ . We find that for *any* combination the result is an ODE of order 21. This means, that the order-21 differential operator  $L_{21}$  is *irreducible*.

**Remark 9:** A last remark on the irreducibility of the large order linear differential operators  $L_{12}^{(\text{left})}$  and  $L_{21}$  is worthy to be mentioned. The operator  $L_{12}^{(\text{left})}$  has been proved to be irreducible in [3]. We also showed, in [3], that *it is not* a symmetric power, or a symmetric product, of *smaller order operators* (see section 3.1 of [3]). We address the same issue on  $L_{21}$ , which even if it is irreducible, it can well be built from factors of lower order, as a symmetric power. The  $n^{\text{th}}$  symmetric power of the generic order- $q$  operator  $L_q$  is  $(q+n-1)!/(n!(q-1)!)$ . For operators  $L_{21}$  and  $\text{Sym}^n(L_q)$  to be equivalent, where the doublet  $(q, n)$  are in the only possibilities  $(2, 20)$ ,  $(3, 5)$  and  $(6, 2)$ , their singular behavior at any singular point should match. The linear differential operator  $L_{21}$  has the same structure of solutions as the operator  $L_{23}$  (see section 4.3 of [4]), except of one analytical (at the origin) and one logarithmic solutions which are solutions of the right factor  $\tilde{L}_2$ . The local structure of the formal solutions (around the origin) of  $L_{21}$  can be grouped as the following. There are two sets of five solutions, behaving as  $\ln(x)^k$ ,  $k = 0, \dots, 4$ , for each set. There are three sets of three solutions behaving as  $\ln(x)^k$ ,  $k = 0, \dots, 2$ , for each set. Finally, two non-logarithmic solutions behaving as  $x^{-11/2}$  and  $x^{-13/2}$ . For the doublets  $(q, n)$ , there is no possibility to obtain *two solutions* with  $\ln(x)^4$ . The linear differential operator  $L_{21}$  is *not a symmetric power of an operator of smaller order*.

## 7. Conclusion

We have shown that the globally nilpotent  $G$ -operators corresponding to the small order ( $\leq 6$ ) factor operators of the linear differential operators annihilating the multifold integrals  $\chi^{(n)}$ , associated with the  $n$ -particle contributions of the magnetic susceptibility of the Ising model ( $n \leq 6$ ), are *homomorphic to their adjoint*. This “duality” property of being self-adjoint up to operator homomorphisms, is equivalent to the fact that their symmetric (or exterior) square have rational solutions [15]. These operators are in selected differential Galois groups like  $SO(q, \mathbb{C})$  and  $Sp(q, \mathbb{C})$ . This self-adjoint (up to operator equivalence) property means that the factor operators, we already know to be Derived from Geometry, are “special” globally nilpotent operators: they correspond to “Special Geometries”.

Two large order operators occur in the factorization of the linear differential operators associated to  $\chi^{(5)}$  and  $\chi^{(6)}$ . The order-twelve operator  $L_{12}^{\text{left}}$  has an *exterior square* that annihilates a rational solution, and the order-21 operator  $L_{21}$  has a *symmetric square* which annihilates a rational solution. The different differential Galois groups are respectively in the symplectic group  $Sp(12, \mathbb{C})$  and the orthogonal group  $SO(21, \mathbb{C})$ .

The two properties (homomorphism with the adjoint and occurrence of a rational solution for the symmetric, or exterior, square), should be verified for these large order operators  $L_{12}^{\text{left}}$  and  $L_{21}$ . Unfortunately, seeking for an homomorphism between these operators, and the corresponding adjoint, is well beyond the possibility of the present

computer facilities. One may just imagine that this can be doable with dedicated programs, computing the homomorphism $\dagger$  modulo primes, with the knowledge that the coefficient of the higher derivative of the intertwiner is $\dagger$  the rational solution we obtained (see (44) and (54)). In view of the results of Appendix C, of many examples, and of the examples on the five-dimensional (and six-dimensional) face-centered cubic lattice Green function [20, 21] (namely  $G_6^{5Dfcc}$  and  $G_8^{6Dfcc}$  in [15], see the intertwiners in equations (40) and (60) in [15]), one may concentrate on intertwiners with *even* orders. It is thus challenging to obtain the intertwiners occurring in the homomorphisms of  $L_{12}^{\text{left}}$  and  $L_{21}$  with their corresponding adjoint, and see whether a "decomposition" (see equation (68) in [15]) in terms of the intertwiners occur, the "decomposition" being probably more complex, but depending on the parity of the order.

Without waiting this "consolidation" of the rational solutions for the symmetric and exterior square of  $L_{12}^{\text{left}}$  and  $L_{21}$ , in terms of homomorphisms of these operators with their adjoint, we can conclude on the intriguing selected character of the (minimal order) globally nilpotent operators annihilating the  $n$ -fold integrals of the Ising model which are all *diagonals of rational functions* [13], and we can conjecture that *all* their operator factors *correspond to selected differential Galois groups*.

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## Appendix A. Miscellaneous polynomials occurring in the main formulae of the paper.

### Appendix A.1. The polynomials $A_{37}(F_3)$ , $P_{34}$ and $P_{53}$ for $F_3$

The apparent polynomial  $A_{37}(F_3)$  of the (monic) order-three operator  $F_3$ , occurs in the expression of the rational solution (16) of the symmetric square of the adjoint of (the monic order-three operator)  $F_3$ , as well as in (29). This degree-27 polynomial reads:

$$\begin{aligned} A_{37}(F_3) = & 5629499534213120 x^{37} + 5348024557502464 x^{36} - 62874472922742784 x^{35} \\ & + 339080589913096192 x^{34} + 132348214635397120 x^{33} + 354600746294968320 x^{32} \\ & + 1383732497338073088 x^{31} - 269118080922157056 x^{30} - 1021414905992970240 x^{29} \\ & + 401943021895024640 x^{28} + 378516473892569088 x^{27} - 379126125978189824 x^{26} \\ & - 181955521970962432 x^{25} + 182991453503356928 x^{24} + 119809766351437824 x^{23} \\ & - 34528714733649920 x^{22} - 46719523456286720 x^{21} - 1865897472688128 x^{20} \end{aligned}$$

$\dagger$  For such "massive" formal calculations switching to the linear differential systems associated with these operators, is probably a way to calculate these homomorphisms. One first obtains the rational solutions of the symmetric or exterior square of these differential systems. The intertwiners are, then, deduced from these rational solutions.

$\ddagger$  See, for instance, relations (C.4) and (C.14).

$$\begin{aligned}
& + 9861412040736768 x^{19} + 1690374175916032 x^{18} - 1285664678690816 x^{17} \\
& - 304716171767808 x^{16} + 112170181177344 x^{15} + 30517814178816 x^{14} \\
& - 7815766123264 x^{13} - 2274047571904 x^{12} + 456062896896 x^{11} + 150282885872 x^{10} \\
& - 10690267808 x^9 - 6048942832 x^8 - 486602112 x^7 + 33772908 x^6 + 25075632 x^5 \\
& + 4670454 x^4 + 13440 x^3 - 69066 x^2 - 5169 x - 63.
\end{aligned} \tag{A.1}$$

A degree-34 polynomial  $P_{34}$  takes place in the expression (11) of the rational solution of the symmetric square of the order-three operator  $F_3$ . This polynomial  $P_{34}$  reads:

$$\begin{aligned}
P_{34}(x) = & 17592186044416 x^{34} - 8796093022208 x^{33} + 204509162766336 x^{32} \\
& - 240793046482944 x^{31} + 347033357516800 x^{30} - 356447925829632 x^{29} \\
& + 307648507412480 x^{28} + 1547605565767680 x^{27} - 1478894410530816 x^{26} \\
& - 3440457380003840 x^{25} + 451333349965824 x^{24} + 3747745613479936 x^{23} \\
& + 2236072096432128 x^{22} - 31693519978496 x^{21} - 472806540705792 x^{20} \\
& - 202845119840256 x^{19} - 55945141010432 x^{18} - 6522043670528 x^{17} \\
& + 8027346038784 x^{16} + 5016548481024 x^{15} + 1158549638912 x^{14} \\
& - 15663757696 x^{13} - 149163564992 x^{12} - 59735088608 x^{11} - 2074333552 x^{10} \\
& + 4173311968 x^9 + 738617492 x^8 - 85245032 x^7 - 26786428 x^6 + 581796 x^5 \\
& + 383308 x^4 - 20652 x^3 - 4867 x^2 + 338 x + 49.
\end{aligned} \tag{A.2}$$

A degree-53 polynomial  $P_{53}$  takes place in the expression (16) of the rational solution of the symmetric square of the adjoint of the (monic) order-three operator  $F_3$ . This polynomial  $P_{53}$  reads:

$$\begin{aligned}
P_{53}(x) = & 5902958103587056517120 x^{53} + 4722366482869645213696 x^{52} \\
& + 135675802662133752135680 x^{51} + 36533776637981766975488 x^{50} \\
& - 60743975313220946624512 x^{49} + 3954166813899570825658368 x^{48} \\
& + 96486199280696075223040 x^{47} - 869025933168471881809920 x^{46} \\
& + 17891408360681834540957696 x^{45} + 19134090460943456531382272 x^{44} \\
& - 6556910656212804697063424 x^{43} - 18888872271338563015016448 x^{42} \\
& - 9641687070213801940877312 x^{41} - 856236460709396327956480 x^{40} \\
& + 1442025047697796450222080 x^{39} + 3004016932710650818330624 x^{38} \\
& + 2353865809090149001199616 x^{37} - 846024305296182175858688 x^{36} \\
& - 1509872584625178282033152 x^{35} + 296963035049372304801792 x^{34} \\
& + 832265748859080390213632 x^{33} + 79111183944514552201216 x^{32} \\
& - 245083727451855922397184 x^{31} - 73512832264242582257664 x^{30} \\
& + 31367300451777147568128 x^{29} + 12977767646670109016064 x^{28} \\
& - 3680779152078761099264 x^{27} - 1138134172734191566848 x^{26} \\
& + 939259567872233308160 x^{25} + 292487910921964093440 x^{24} \\
& - 129084866249262874624 x^{23} - 72153802925319249920 x^{22} \\
& + 464226011870542848 x^{21} + 8000870954669244416 x^{20} \\
& + 1689242686839294720 x^{19} - 289875180323084800 x^{18}
\end{aligned}$$

$$\begin{aligned}
& -171427111790469312x^{17} - 17240190260449408x^{16} + 4666400462438480x^{15} \\
& + 1816703798900448x^{14} + 258529109814976x^{13} - 29106463737504x^{12} \\
& - 20951763420448x^{11} - 2341127444328x^{10} + 460438019724x^9 \\
& + 115534150804x^8 - 15491040x^7 - 1792901976x^6 - 94207344x^5 \\
& + 6320658x^4 - 571740x^3 - 192705x^2 - 10869x - 147.
\end{aligned} \tag{A.3}$$

*Appendix A.2. The polynomial  $P_{10}$  for  $\tilde{L}_3$*

A degree-ten polynomial  $P_{10}$  takes place in the expression of the rational solution (22) of the symmetric square of the adjoint of the (monic) order-three operator  $\tilde{L}_3$ . This polynomial  $P_{10}$  reads:

$$\begin{aligned}
P_{10}(x) = & 19394461696x^{10} - 17411604480x^9 + 6106742784x^8 - 1095237312x^7 \\
& + 158668656x^6 - 36766920x^5 + 7627535x^4 - 900594x^3 \\
& + 57342x^2 - 1856x + 24.
\end{aligned} \tag{A.4}$$

*Appendix A.3. The polynomials  $A_{26}(L_4)$  and  $P_{17}$  for  $L_4$*

The apparent polynomial  $A_{26}(L_4)$  of the order-four operator  $L_4$ , occurs in the expression of the rational solution (25) of the exterior square of the adjoint of (the monic order-four operator)  $L_4$ , as well as in the order-two intertwiner (31). This degree-26 polynomial reads:

$$\begin{aligned}
A_{26}(L_4) = & 521686412421099571093753036800x^{26} \\
& - 724445324775545659452335063040x^{25} + 45081769872830521912080728064x^{24} \\
& + 616797192523902897669611192320x^{23} - 636026962079787427490890252288x^{22} \\
& + 359505820412663945726355570688x^{21} - 142807225508285034141616963584x^{20} \\
& + 43345424617004971574289235968x^{19} - 10332892566359614848157876224x^{18} \\
& + 1953967934450852091348254720x^{17} - 296338746597146803591135232x^{16} \\
& + 36761552740911534545901568x^{15} - 3850023960384577768909952x^{14} \\
& + 354446803792968575565792x^{13} - 29645475671183771992224x^{12} \\
& + 2252938824290334087840x^{11} - 148605250583921845896x^{10} \\
& + 7727889974481947660x^9 - 251264549473230968x^8 - 1305110830870633x^7 \\
& + 766418384173454x^6 - 52582954690298x^5 + 2099285510560x^4 \\
& - 54037012120x^3 + 873083400x^2 - 7854000x + 2800.
\end{aligned} \tag{A.5}$$

A degree-17 polynomial  $P_{17}$  takes place in the expression of the homomorphism (29) of  $\tilde{L}_3$  with its adjoint, as well as in the rational solution (24) of the exterior square of  $L_4$  and the rational solution (25) of the exterior square of the adjoint of (the monic order-four operator)  $L_4$ . This polynomial  $P_{17}$  reads:

$$\begin{aligned}
P_{17}(x) = & 140082179425173504x^{17} - 496507256028790784x^{16} \\
& + 705909942330064896x^{15} - 440315308230574080x^{14} \\
& + 141123001405931520x^{13} - 25595376023494656x^{12} + 4059589860750336x^{11}
\end{aligned}$$

$$\begin{aligned}
& -1133589089074624 x^{10} + 350453101085400 x^9 - 74115473257440 x^8 \\
& + 10126459925120 x^7 - 904049598675 x^6 + 52738591890 x^5 - 1959091320 x^4 \\
& + 43407720 x^3 - 502593 x^2 + 2548 x - 12.
\end{aligned} \tag{A.6}$$

## Appendix B. Some linear differential operators appearing in Section 3

### Appendix B.1. Linear differential operators for $F_3$

The order-two differential operator  $R_2$  occurring in the solution of  $F_3$  given in (15) reads

$$\begin{aligned}
R_2 = & -\frac{x \cdot (1 - 4x^2)(1 - 16x^2) \cdot P_{15}(x)}{D_5(x) \cdot A_7(F_2)} \cdot D_x^2 \\
& - \frac{P_{20}(x)}{D_5(x) \cdot A_7(F_2)} \cdot D_x + \frac{8x \cdot P_{22}(x)}{D_5(x) \cdot A_7(F_2)},
\end{aligned}$$

where  $A_7(F_2)$  is the apparent polynomial of  $F_2$  given in (12), and where:

$$\begin{aligned}
D_5(x) &= x \cdot (1 - x)(1 + 2x)(4x^2 + 3x + 1), \\
P_{15}(x) &= 2 + 9x - 99x^2 - 873x^3 - 1865x^4 + 12140x^5 + 83412x^6 \\
&+ 238912x^7 + 375008x^8 - 1397504x^9 - 9548288x^{10} - 17188864x^{11} \\
&- 7581696x^{12} + 2260992x^{13} - 7471104x^{14} - 786432x^{15}, \\
P_{22}(x) &= 2 - 5x - 233x^2 + 43x^3 + 7343x^4 + 14408x^5 + 28660x^6 \\
&+ 68224x^7 - 2196448x^8 - 13292608x^9 - 21440000x^{10} + 94341632x^{11} \\
&+ 562065408x^{12} + 700620800x^{13} - 1591803904x^{14} - 4451794944x^{15} \\
&- 2017984512x^{16} + 467664896x^{17} - 2013265920x^{18} - 268435456x^{19} \\
&+ 67108864x^{20}, \\
P_{18}(x) &= 4 + 13x + 87x^2 + 1472x^3 - 1950x^4 - 34896x^5 + 53220x^6 \\
&+ 630696x^7 + 536416x^8 - 4436416x^9 - 21416192x^{10} - 32954368x^{11} \\
&+ 80510976x^{12} + 304701440x^{13} + 227115008x^{14} + 5636096x^{15} \\
&+ 151519232x^{16} + 29360128x^{17} - 16777216x^{18}.
\end{aligned}$$

### Appendix B.2. Linear differential operators for $\tilde{L}_3$

The order-two linear differential operator  $R_2$  occurring in the solution of  $\tilde{L}_3$  given in (21) is

$$\begin{aligned}
R_2 = & \frac{x^2 \cdot Q_3(x) \cdot P_4(x)}{(1 - 4x)(1 - 16x)^7} \cdot D_x^2 + \frac{x \cdot Q_3(x)P_6(x)}{(1 - 4x)^2(1 - 16x)^8} \cdot D_x \\
& + \frac{216x^3 \cdot (1 - 12x) \cdot Q_3(x) \cdot P_3(x)}{(1 - 4x)^3(1 - 16x)^9},
\end{aligned} \tag{B.1}$$

with:

$$\begin{aligned}
Q_3 &= 5 - 160x + 1232x^2 - 1024x^3, \\
P_4 &= 3 - 68x + 976x^2 - 2624x^3 - 61440x^4, \\
P_6 &= 3 - 152x + 3528x^2 - 42384x^3 + 89024x^4 + 1966080x^5 - 5898240x^6, \\
P_3 &= 3 - 92x + 1792x^2 + 4096x^3.
\end{aligned}$$

Appendix B.3. The linear differential operator  $L_4$ 

The linear differential operator  $L_4$  has been analyzed in [6] and was shown to be equivalent to a *Calabi-Yau equation* with solution

$${}_4F_3\left(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], [1, 1, 1]; z\right), \quad (\text{B.2})$$

where the argument  $z$  is an *algebraic pullback*, in the variable  $w = s/2/(1+s^2)$ , (with  $s = \sinh(2K)$ , where  $K = J/kT$  is the Ising model coupling constant):

$$z = \left(\frac{1 + (1 - 16 \cdot w^2)^{1/2}}{1 - (1 - 16 \cdot w^2)^{1/2}}\right)^4 = s^8. \quad (\text{B.3})$$

Note that the variable  $w$  deals equally with the high and low regime of temperature. One has another pullback which is  $1/z = 1/s^8$ .

Appendix B.3.1. Direct sum structure associated with  $L_4$ 

When written in the variable  $x = w^2$ , the  ${}_4F_3$  hypergeometric function *with any of the two pullbacks*, for instance the series with *integer coefficients*

$${}_4F_3\left(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], [1, 1, 1]; \frac{1}{z}\right) = 1 + 16x^4 + 512x^5 + 11264x^6 + \dots, \quad (\text{B.4})$$

is annihilated by an *order-eight* linear operator  $L_8 = H_4^{(1)} \oplus H_4^{(2)}$  which is a *direct sum of two order-four* linear differential operators  $H_4^{(1)}$  and  $H_4^{(2)}$ :

$$\begin{aligned} H_4^{(1)} = & x^3 \cdot (1 - 16x)^2 (1 - 8x)^4 \cdot D_x^4 \\ & + 2x^2 \cdot (1 - 8x)^3 (1 - 16x) (512x^2 - 96x + 3) \cdot D_x^3 \\ & + x \cdot (1 - 8x)^2 (233472x^4 - 83968x^3 + 10880x^2 - 512x + 7) \cdot D_x^2 \\ & - (1 - 8x) (589824x^5 - 266240x^4 + 40960x^3 - 3968x^2 + 144x - 1) \cdot D_x \\ & - 256x \cdot (1 - 16x - 128x^2), \end{aligned}$$

$$\begin{aligned} H_4^{(2)} = & x^3 \cdot (1 - 16x)^4 (1 - 8x) \cdot D_x^4 \\ & + 2x^2 \cdot (1 - 16x)^3 (640x^2 - 96x + 3) \cdot D_x^3 \\ & - x \cdot (1 - 16x)^2 (50688x^3 - 8768x^2 + 456x - 7) \cdot D_x^2 \\ & + (1 - 16x) (466944x^4 - 90112x^3 + 5952x^2 - 144x + 1) \cdot D_x \\ & + 256x \cdot (1 - 8x) (192x^2 - 16x + 1). \end{aligned}$$

Each one corresponds to a Calabi-Yau ODE [6].

Note that these two order-four operators are simply conjugated:

$$\frac{\sqrt{1-16x}}{1-8x} \cdot H_4^{(1)} = H_4^{(2)} \cdot \frac{\sqrt{1-16x}}{1-8x}. \quad (\text{B.5})$$

The solution of  $H_4^{(2)}$  analytic at  $x = 0$  is the series with *integer coefficients*:

$$\begin{aligned} \text{sol}(H_4^{(2)}) &= \frac{1}{4} \cdot \frac{\sqrt{1-16x}}{x \cdot z^{1/4}} \cdot {}_4F_3\left(\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right], [1, 1, 1]; \frac{1}{z}\right) \\ &= 1 - 16x^2 - 256x^3 - 3568x^4 - 48640x^5 - 664832x^6 + \dots \end{aligned} \quad (\text{B.6})$$

The solution of  $H_4^{(1)}$  analytic at  $x = 0$  is the series with *integer coefficients*:

$$\begin{aligned} \text{sol}(H_4^{(1)}) &= 1 + 16x^2 + 256x^3 + 3600x^4 + 49664x^5 + 687360x^6 \\ &+ 9596928x^7 + 135300368x^8 + 1925268480x^9 + \dots \end{aligned} \quad (\text{B.7})$$

The simple  ${}_4F_3$  hypergeometric function (B.4) is actually equal to the half sum  $(\text{sol}(H_4^{(1)}) + \text{sol}(H_4^{(2)}))/2$ , of the two solutions (B.6) and (B.7) of  $H_4^{(2)}$  and  $H_4^{(1)}$ .

### Appendix B.3.2. Solution of the linear differential operators for $L_4$

The order-four operator  $H_4^{(2)}$  is homomorphic to the order-four operator  $L_4$ , emerging as a factor operator for  $\tilde{\chi}^{(6)}$ :

$$S_3 \cdot H_4^{(2)} = L_4 \cdot R_3, \quad (\text{B.8})$$

where  $S_3$  and  $R_3$  are two order-three intertwiners. One immediately deduces the solution of  $L_4$  given in terms of the intertwiner  $R_3$  in (B.8) acting on the solution of the order-four operator  $H_4^{(2)}$ :

$$\text{sol}(L_4) = R_3 \left( \text{sol}(H_4^{(2)}) \right), \quad (\text{B.9})$$

where the order-three linear differential operator  $R_3$  reads:

$$\begin{aligned} & 36x^5 \cdot (1-4x)(1-16x)^{11} \cdot A_4(\tilde{L}_3) \cdot R_3 \\ &= -x^2 \cdot (1-8x)(1-16x)^6 \cdot Q_3 \cdot D_x^3 - x(1-16x)^5 \cdot Q_2 \cdot D_x^2 \\ & \quad - (1-16x)^4 \cdot Q_1 \cdot D_x - 128x \cdot (1-16x)^3 \cdot Q_0, \quad (\text{B.10}) \\ Q_3 &= 20 - 2270x + 106086x^2 - 2675757x^3 + 40471555x^4 - 389549218x^5 \\ & \quad + 2566958582x^6 - 13288554644x^7 + 53910201600x^8 - 95886464512x^9 \\ & \quad - 40752267264x^{10} + 93413441536x^{11} + 82141249536x^{12}, \\ Q_2 &= 60 - 8730x + 551602x^2 - 19952295x^3 + 459567769x^4 - 7113445902x^5 \\ & \quad + 76621809730x^6 - 596173812436x^7 + 3524748623424x^8 \\ & \quad - 16119878544384x^9 + 49591145041920x^{10} - 62942370168832x^{11} \\ & \quad - 43186282037248x^{12} + 62103616487424x^{13} + 63084479643648x^{14}, \\ Q_1 &= 20 - 3710x + 303254x^2 - 14374525x^3 + 439222171x^4 - 9126353218x^5 \\ & \quad + 133114097446x^6 - 1396508587356x^7 + 10831258373280x^8 \\ & \quad - 63997739175680x^9 + 285429913462784x^{10} - 832850214682624x^{11} \\ & \quad + 969294168981504x^{12} + 842807128358912x^{13} - 1089827550265344x^{14} \\ & \quad - 1135520633585664x^{15}, \\ Q_0 &= 20 - 2590x + 145574x^2 - 4725757x^3 + 99952043x^4 - 1473719054x^5 \\ & \quad + 15848325886x^6 - 128583477160x^7 + 795236207808x^8 - 3570673925376x^9 \\ & \quad + 9940600639488x^{10} - 10105313820672x^{11} - 13061917245440x^{12} \\ & \quad + 14868774125568x^{13} + 15771119910912x^{14}, \end{aligned}$$

the apparent polynomial  $A_4(\tilde{L}_3)$  being given in (23).

## Appendix C. Homomorphism with the adjoint for order-three and order-four operators

We show here, starting with *generic* (and irreducible) operators, the link between the homomorphism with the adjoint and the occurrence of a rational solution of the symmetric (or exterior) square of the differential operator for operators of order three and four.

For a linear differential operator of order  $q$ , the order of the intertwiner in an equivalence relation may reach the order  $q - 1$ . Appendix C.1 considers order-three generic operator with intertwiner of order two, one and zero (i.e. a function). Appendix C.2 deals with order-four generic operator with order-two and order-zero intertwiner, and Appendix C.3 is for the case of order-three and order-one intertwiner.

*Appendix C.1. Order-three linear differential operator*

With the generic order-three differential operator  $L_3$

$$L_3 = D_x^3 + p_2(x) \cdot D_x^2 + p_1(x) \cdot D_x + p_0(x), \quad (\text{C.1})$$

and the order-two differential operator

$$R_2 = a_2(x) \cdot D_x^2 + a_1(x) \cdot D_x + a_0(x), \quad (\text{C.2})$$

one demands that the relation

$$L_3 \cdot R_2 = \text{adjoint}(R_2) \cdot \text{adjoint}(L_3), \quad (\text{C.3})$$

be fulfilled, which means that  $L_3$  is homomorphic to its adjoint.

Zeroing the expressions in front of each derivative  $D_x^j$  in (C.3) gives a set of equations which solve as

$$a_2(x) = \text{sol}(\text{Sym}^2(L_3)) \quad (\text{C.4})$$

$$a_1(x) = -p_2(x) \cdot a_2(x) - \frac{1}{2} \frac{da_2(x)}{dx}, \quad (\text{C.5})$$

and

$$a_0(x) = N_5 \cdot a_2(x). \quad (\text{C.6})$$

The order-five differential operator  $N_5$  is such that

$$C_1^{(3)} \cdot N_5 = 9 D_x^5 + 30 p_2(x) \cdot D_x^4 + Q_3 \cdot D_x^3 + Q_2 \cdot D_x^2 + Q_1 \cdot D_x + Q_0, \quad (\text{C.7})$$

where

$$\begin{aligned} Q_3 &= 25 p_2(x)^2 + 45 p_1(x) + 15 \frac{dp_2(x)}{dx}, \\ Q_2 &= 75 p_1(x) p_2(x) + 45 p_0(x) + 45 \frac{dp_1(x)}{dx}, \\ Q_1 &= 36 p_1(x)^2 - 4 p_2(x)^4 + 42 p_2(x) p_0(x) + 22 p_2(x)^2 p_1(x) \\ &\quad + 9 \frac{d^2 p_1(x)}{dx^2} + 63 \frac{dp_0(x)}{dx} - 18 \frac{dp_2(x)}{dx} p_2(x)^2 + 48 p_2(x) \frac{d}{dx} p_1(x) \\ &\quad - 3 p_1(x) \frac{dp_2(x)}{dx} - 9 p_2(x) \frac{d^2 p_2(x)}{dx^2}, \\ Q_0 &= 36 p_1(x)^2 p_2(x) - 36 p_1(x) p_0(x) - 8 p_1(x) p_2(x)^3 + 8 p_2(x)^2 p_0(x) \\ &\quad - 18 \frac{d^2 p_2(x)}{dx^2} p_1(x) + 18 \frac{d^2 p_2(x)}{dx^2} \frac{dp_2(x)}{dx} + 36 \left( \frac{d}{dx} p_2(x) \right)^2 p_2(x) \\ &\quad + 102 \frac{dp_2(x)}{dx} p_0(x) - 54 \frac{dp_2(x)}{dx} \frac{dp_1(x)}{dx} + 8 \frac{dp_2(x)}{dx} p_2(x)^3 + 18 \frac{d^2 p_0(x)}{dx^2} \\ &\quad + 42 p_2(x) \frac{dp_0(x)}{dx} + 54 p_1(x) \frac{dp_1(x)}{dx} - 72 \frac{dp_2(x)}{dx} p_1(x) p_2(x), \end{aligned}$$

and

$$\begin{aligned} C_1^{(3)} &= 36 p_1(x) p_2(x) - 108 p_0(x) - 8 p_2(x)^3 - 36 p_2(x) \frac{dp_2(x)}{dx} \\ &\quad + 54 \frac{dp_1(x)}{dx} - 18 \frac{d^2 p_2(x)}{dx^2}. \end{aligned}$$

The last expression, when  $C_1^{(3)} = 0$ , is the condition for  $L_3$  to have an order-five symmetric square instead of the order six.



From the expression (C.4), one sees that if  $Sym^2(L_3)$  has a *rational solution*, one may take  $a_2(x)$  as *this solution*, and automatically the expressions of  $a_1(x)$  and  $a_0(x)$  will be rational. If  $Sym^2(L_3)$  has no rational solution (assume  $Sym^2(L_3)$  is irreducible), one may still take for  $a_2(x)$  *any solution* of  $Sym^2(L_3)$  and with the corresponding  $a_1(x)$ ,  $a_0(x)$ , the relation (C.3) will be verified. But now, the intertwiner  $R_2$  is *no more over the rationals*.

Note that, one may use (C.4) and (C.6) with its derivative, to reduce (C.6) to the following inhomogeneous differential equation

$$6 \frac{da_0(x)}{dx} + 4p_2(x) \cdot a_0(x) = E_3 \cdot a_2(x) \quad (C.8)$$

where the order-three differential operator  $E_3$  is

$$\begin{aligned} E_3 = & D_x^3 + 3p_2(x) \cdot D_x^2 + \left(4p_1(x) + 2p_2(x)^2 - 3 \frac{dp_2(x)}{dx}\right) \cdot D_x \\ & + 4p_1(x)p_2(x) - 4p_0(x) - 6 \frac{d^2p_2(x)}{dx^2} + 6 \frac{dp_1(x)}{dx} - 4p_2(x) \frac{dp_2(x)}{dx}. \end{aligned}$$

The way to obtain  $a_0(x)$ , via (C.6), is more tractable, since this amounts to taking derivatives of the rational solution  $a_2(x)$ . The route via (C.8) calls for an integration, and re-injection in (C.3), to fix the constants of integration.

Instead of an intertwiner  $R_2$  of order two, let us consider the situation with an order-one intertwiner,  $a_1(x) \cdot D_x + a_0(x)$ . In this case, one obtains

$$a_0(x) = -p_2(x) \cdot a_1(x) - \frac{da_1(x)}{dx} \quad \text{and:} \quad a_1(x) = \text{sol}(Ext^2(L_3)). \quad (C.9)$$

Recall that (with  $W_L(x)$  the Wronskian of  $L_3$ )

$$Ext^2(L_3) \cdot W_L(x) = W_L(x) \cdot \text{adjoint}(L_3). \quad (C.10)$$

Since  $L_3$  is irreducible,  $a_1(x)$  cannot be rational. Therefore, there is no homomorphism between  $L_3$  and its adjoint with an order-one intertwiner over the rationals. For order zero intertwiner, i.e. a function  $a_0(x)$ , one obtains

$$a_0(x) = W_L(x)^{2/3} \quad \text{and} \quad C_1^{(3)} = 0, \quad (C.11)$$

where  $W_L(x)$  is the Wronskian of  $L_3$ . The condition  $C_1^{(3)} = 0$  makes the symmetric square of  $L_3$  of order five, and  $L_3$  is the symmetric square of an order-two differential operator.

### Appendix C.2. Order-four linear differential operator

What we have done for the generic order-three linear differential operator can be repeated for a generic order-four operator  $L_4$ .

For the order-four differential operator

$$L_4 = D_x^4 + p_3(x) \cdot D_x^3 + p_2(x) \cdot D_x^2 + p_1(x) \cdot D_x + p_0(x), \quad (C.12)$$

and an order-two operator as in (C.2), the relation

$$L_4 \cdot R_2 = \text{adjoint}(R_2) \cdot \text{adjoint}(L_4), \quad (C.13)$$

is solved to give

$$a_2(x) = \text{sol}(Ext^2(L_4)), \quad a_1(x) = -p_3(x) \cdot a_2(x) - \frac{da_2(x)}{dx}, \quad (C.14)$$

and

$$a_0(x) = N_5 \cdot a_2(x). \quad (\text{C.15})$$

The order-five differential operator  $N_5$  is such that

$$C_1^{(4)} \cdot N_5 = 4D_x^5 + 10p_3(x) \cdot D_x^4 + Q_3 \cdot D_x^3 + Q_2 \cdot D_x^2 + Q_1 \cdot D_x + Q_0, \quad (\text{C.16})$$

where

$$\begin{aligned} Q_3 &= 7(p_3(x))^2 + 8p_2(x) + 8\frac{dp_3(x)}{dx}, \\ Q_2 &= 14p_2(x)p_3(x) - 4p_1(x) + 16\frac{dp_2(x)}{dx}, \\ Q_1 &= 4p_2(x)^2 - 16p_0(x) + 5p_3(x)^2p_2(x) - 2p_3(x)p_1(x) - p_3(x)^4 \\ &\quad + 8\frac{dp_1(x)}{dx} + 4\frac{d^2p_2(x)}{dx^2} - 6p_3(x)^2\frac{dp_3(x)}{dx} + 14p_3(x)\frac{dp_2(x)}{dx} \\ &\quad - 4p_3(x)\frac{d^2p_3(x)}{dx^2}, \\ Q_0 &= 4p_3(x)p_2(x)^2 - p_2(x)p_3(x)^3 + p_3(x)^2p_1(x) - 8p_0(x)p_3(x) \\ &\quad - 4p_2(x)p_1(x) + 4\frac{dp_3(x)}{dx}\frac{d^2p_3(x)}{dx^2} - 4p_2(x)\frac{d^2p_3(x)}{dx^2} + 4\frac{d^2p_1(x)}{dx^2} \\ &\quad + 6\left(\frac{dp_3(x)}{dx}\right)^2p_3(x) + p_3(x)^3\frac{dp_3(x)}{dx} - 10p_3(x)p_2(x)\frac{dp_3(x)}{dx} \\ &\quad + 8\frac{dp_3(x)}{dx}p_1(x) - 8\frac{dp_3(x)}{dx}\frac{dp_2(x)}{dx} + 8p_2(x)\frac{dp_2(x)}{dx} \\ &\quad + 6p_3(x)\frac{dp_1(x)}{dx} - 8\frac{dp_0(x)}{dx}, \end{aligned}$$

and:

$$\begin{aligned} C_1^{(4)} &= 4p_2(x)p_3(x) - 8p_1(x) - p_3(x)^3 + 8\frac{dp_2(x)}{dx} - 4\frac{d^2p_3(x)}{dx^2} \\ &\quad - 6p_3(x)\frac{dp_3(x)}{dx}. \end{aligned} \quad (\text{C.17})$$

Here also,  $C_1^{(4)} = 0$  is the so-called [15] “Calabi-Yau condition”. When verified,  $Ext^2(L_4)$  is of order five, instead of the order six.

All what have been said for  $L_3$  (on the rationality of the coefficients) holds. In particular, (C.15) can be reduced to the inhomogeneous linear differential equation

$$2\frac{da_0(x)}{dx} + p_3(x) \cdot a_0(x) = E_3 \cdot a_2(x) \quad (\text{C.18})$$

with:

$$\begin{aligned} E_3 &= D_x^3 + 2p_3(x) \cdot D_x^2 + (p_3(x)^2 + p_2(x)) \cdot D_x \\ &\quad + p_2(x)p_3(x) - p_1(x) + 2\frac{dp_2(x)}{dx} - 2\frac{d^2p_3(x)}{dx^2} - p_3(x)\frac{dp_3(x)}{dx}. \end{aligned}$$

Similarly to  $L_3$ , one may consider an order-zero intertwiner  $a_0(x)$  for  $R_2$ . In this case, one obtains

$$a_0(x) = W_{L_4}(x)^{1/2} \quad \text{and} \quad C_1^{(4)} = 0. \quad (\text{C.19})$$

$W_{L_4}(x)$  is the Wronskian of  $L_4$  and  $C_1^{(4)} = 0$  is the Calabi-Yau condition [15] given in (C.17), which if fulfilled,  $L_4$  has an order-five exterior square instead of order six.

Appendix C.3. Order-four differential operator and  $SO(4, \mathbb{C})$ 

We consider the equivalence relation

$$L_4 \cdot R_3 = \text{adjoint}(R_3) \cdot \text{adjoint}(L_4), \quad (\text{C.20})$$

with  $R_3$  of order three

$$R_3 = a_3(x) \cdot D_x^3 + a_2(x) \cdot D_x^2 + a_1(x) \cdot D_x + a_0(x). \quad (\text{C.21})$$

One obtains

$$a_3(x) = \text{sol}(\text{Sym}^2(L_4)), \quad a_2(x) = -p_3(x) \cdot a_3(x) - \frac{1}{2} \frac{da_3(x)}{dx}. \quad (\text{C.22})$$

The coefficient  $a_1(x)$  is a solution of the order-three inhomogeneous differential equation  $\ddagger$

$$N_3 \cdot a_1(x) = N_5 \cdot a_3(x), \quad (\text{C.23})$$

where the order-three operator  $N_3$  is

$$\begin{aligned} N_3 = & 5 D_x^3 + \frac{15}{2} p_3(x) \cdot D_x^2 + \left( \frac{9}{2} \frac{dp_3(x)}{dx} + 3 p_3(x)^2 + 2 p_2(x) \right) \cdot D_x \\ & + \left( 3 \frac{dp_2(x)}{dx} - 2 p_1(x) + 2 p_2(x) p_3(x) \right), \end{aligned} \quad (\text{C.24})$$

and the order-five operator  $N_5$  reads

$$\begin{aligned} N_5 = & D_x^5 + \frac{15}{4} p_3(x) \cdot D_x^4 + \left( \frac{17}{4} p_3(x)^2 - \frac{7}{4} \frac{dp_3(x)}{dx} + \frac{9}{2} p_2(x) \right) \cdot D_x^3 \\ & + Q_2 \cdot D_x^2 + Q_1 \cdot D_x + Q_0, \end{aligned} \quad (\text{C.25})$$

with:

$$\begin{aligned} Q_2 = & \frac{3}{2} p_3(x)^3 + \frac{17}{2} p_2(x) p_3(x) - \frac{5}{2} p_1(x) - \frac{45}{2} \frac{d^2 p_3(x)}{dx^2} \\ & - \frac{21}{4} p_3(x) \frac{dp_3(x)}{dx} + 15 \frac{dp_2(x)}{dx}, \end{aligned} \quad (\text{C.26})$$

$$\begin{aligned} Q_1 = & 2 p_2(x)^2 - 2 p_1(x) p_3(x) + 4 p_2(x) p_3(x)^2 - 4 p_0(x) - \frac{27}{4} \left( \frac{dp_3(x)}{dx} \right)^2 \\ & - \frac{105}{4} p_3(x) \frac{d^2 p_3(x)}{dx^2} - \frac{9}{2} p_3(x)^2 \frac{dp_3(x)}{dx} + \frac{3}{2} p_2(x) \frac{dp_3(x)}{dx} \\ & + \frac{33}{2} p_3(x) \frac{dp_2(x)}{dx} + 15 \frac{d^2 p_2(x)}{dx^2} - \frac{55}{2} \frac{d^3 p_3(x)}{dx^3} - \frac{3}{2} \frac{dp_1(x)}{dx}, \end{aligned} \quad (\text{C.27})$$

$$\begin{aligned} Q_0 = & 2 p_3(x) (p_2(x))^2 - 2 p_0(x) p_3(x) - 2 p_1(x) p_2(x) - 10 \frac{d^4 p_3(x)}{dx^4} \\ & - 3 \frac{dp_0(x)}{dx} - 15 p_3(x) \frac{d^3 p_3(x)}{dx^3} - \frac{3}{2} \frac{dp_2(x)}{dx} \frac{dp_3(x)}{dx} + 5 p_2(x) \frac{d}{dx} p_2(x) \\ & - 9 \frac{dp_3(x)}{dx} \frac{d^2 p_3(x)}{dx^2} + 4 p_1(x) \frac{dp_3(x)}{dx} - 4 p_2(x) \frac{d^2 p_3(x)}{dx^2} + 5 \frac{d^3 p_2(x)}{dx^3} \\ & + \frac{15}{2} p_3(x) \frac{d^2 p_2(x)}{dx^2} - 6 p_3(x)^2 \frac{d^2 p_3(x)}{dx^2} - 4 p_2(x) p_3(x) \frac{dp_3(x)}{dx} \\ & + 3 \frac{dp_2(x)}{dx} p_3(x)^2. \end{aligned} \quad (\text{C.28})$$

$\ddagger$  Note that  $a_1(x)$  is also given by  $a_1(x) = N_9 \cdot a_3(x)$ , where  $N_9$  is an order-nine linear differential operator. Once  $a_3(x)$  is rational,  $a_1(x)$  will be rational.

The coefficient  $a_0(x)$  is given by

$$a_0(x) = -p_3(x) \cdot a_1(x) - \frac{3}{2} \frac{da_1(x)}{dx} + \frac{1}{4} E_3 \cdot a_3(x), \quad (\text{C.29})$$

where the order-three operator  $E_3$  reads:

$$\begin{aligned} E_3 = & D_x^3 + 3p_3(x) \cdot D_x^2 + \left( 4p_2(x) - 7 \frac{dp_3(x)}{dx} + 2p_3(x)^2 \right) \cdot D_x \\ & + 4p_2(x)p_3(x) - 4p_1(x) - 16 \frac{d^2p_3(x)}{dx^2} + 10 \frac{dp_2(x)}{dx} - 8p_3(x) \frac{dp_3(x)}{dx}. \end{aligned} \quad (\text{C.30})$$

For the equivalence, between the differential operator  $L_4$  and its adjoint with an order-three intertwiner *over the rationals*, to exist, it is the *symmetric square* of  $L_4$  that should annihilate a rational solution.

We consider, now, the case of an order-one intertwiner  $R_1 = a_1(x) \cdot D_x + a_0(x)$  in

$$L_4 \cdot R_1 = \text{adjoint}(R_1) \cdot \text{adjoint}(L_4). \quad (\text{C.31})$$

One obtains for  $a_0(x)$

$$a_0(x) = -p_3(x) \cdot a_1(x) - \frac{3}{2} \frac{da_1(x)}{dx}, \quad (\text{C.32})$$

and

$$a_1(x) = \text{sol}(E_2), \quad (\text{C.33})$$

where the order-two linear differential operator  $E_2$  reads:

$$\begin{aligned} E_2 = & C_1^{(4)} \cdot D_x^2 + \frac{1}{105} Q_1 \cdot D_x + \frac{2}{105} Q_0, \quad (\text{C.34}) \\ Q_1 = & 212 p_2(x) p_3(x)^2 - 66 p_3(x)^4 - 440 p_1(x) p_3(x) + 144 p_2(x)^2 - 1600 p_0(x), \\ & + 54 \left( \frac{dp_3(x)}{dx} \right)^2 - 292 p_2(x) \frac{dp_3(x)}{dx} - 60 \frac{d^3 p_3(x)}{dx^3} - 40 \frac{d^2 p_2(x)}{dx^2} \\ & + 520 \frac{dp_1(x)}{dx} + 580 \frac{dp_2(x)}{dx} p_3(x) - 423 p_3(x)^2 \frac{dp_3(x)}{dx} - 390 p_3(x) \frac{d^2 p_3(x)}{dx^2}, \\ Q_0 = & -22 p_2(x) p_3(x)^3 + 22 p_1(x) p_3(x)^2 - 72 p_1(x) p_2(x) + 72 p_2(x)^2 p_3(x) \\ & - 400 p_0(x) p_3(x) - 13 \frac{dp_2(x)}{dx} p_3(x)^2 + 18 \frac{dp_2(x)}{dx} \frac{dp_3(x)}{dx} - 20 \frac{d^3 p_2(x)}{dx^3} \\ & - 50 \frac{d^2 p_2(x)}{dx^2} p_3(x) + 180 \frac{dp_1(x)}{dx} p_3(x) - 52 p_1(x) \frac{dp_3(x)}{dx} \\ & - 128 p_3(x) p_2(x) \frac{dp_3(x)}{dx} - 200 \frac{dp_0(x)}{dx} + 80 \frac{d^2 p_1(x)}{dx^2} - 80 p_2(x) \frac{d^2 p_3(x)}{dx^2} \\ & + 108 \frac{dp_2(x)}{dx} p_2(x). \end{aligned}$$

Here, our analysis is a little bit incomplete. For all the examples so far found, this situation corresponds to a symmetric square of order nine instead of the order ten. We do not know how to prove that (C.34) should have a rational solution in this case. One may just imagine, that this is provable if we use the symmetric Calabi-Yau condition (the corresponding condition for order-four operators that makes the symmetric square of order nine instead of ten) to factorize, or reduce to order one, the operator (C.34). Unfortunately, this condition for order-four operator is very large, and nonlinear, in the coefficients of  $L_4$  and their derivatives: it is sum of 3548 monomials and has degree 12 in the coefficients (and their derivatives) of  $L_4$ .

### Appendix D. Quadratic (or alternating) invariant forms

The differential Galois group of the order-three linear differential operator  $F_3$ , occurring in  $\tilde{\chi}^{(5)}$ , is in the orthogonal group  $SO(3, \mathbb{C})$ . We denote by  $X_0$  any of its formal solutions at  $x = 0$  and by  $X_1$  (and  $X_2$ ) respectively the first (and second) derivative. The first integral of  $F_3$  reads

$$Q(X_0, X_1, X_2) = \text{const.} \quad (\text{D.1})$$

with

$$\begin{aligned} Q(X_0, X_1, X_2) = & \frac{x \cdot (1+4x)^2(1-4x)^6 \cdot P_{101}(x)}{A_7(F_2)^4 \cdot A_{37}(F_3)^2} \cdot X_0^2 \\ & + \frac{x^3 \cdot (1+4x)^4(1-4x)^8(1+2x)^2 \cdot P_{81}(x)}{A_7(F_2)^2 \cdot A_{37}(F_3)^2} \cdot X_1^2 \\ & + \frac{x^5 \cdot (1+4x)^6(1-4x)^{10}(1-x)^2(1+2x)^4(1-2x)^2(1+3x+4x^2)^2 \cdot P_{53}(x)}{A_{37}(F_3)^2} \cdot X_2^2 \\ & - \frac{x^4 \cdot (1+4x)^5(1-4x)^9(1-x)(1+2x)^3(1-2x)(1+3x+4x^2) \cdot P_{67}(x)}{A_7(F_2) \cdot A_{37}(F_3)^2} \cdot X_1 X_2 \\ & + \frac{x^3 \cdot (1+4x)^4(1-4x)^8(1+2x)^2(1-x)(1-2x)(1+3x+4x^2) \cdot P_{77}(x)}{A_7(F_2)^2 \cdot A_{37}(F_3)^2} \cdot X_0 X_2 \\ & - \frac{x^2 \cdot (1+4x)^3(1-4x)^7(1+2x) \cdot P_{91}(x)}{A_7(F_2)^3 \cdot A_{37}(F_3)^2} \cdot X_0 X_1. \end{aligned} \quad (\text{D.2})$$

The  $P_j(x)$ 's are polynomials of degree  $j$ . The numerical value of  $\text{const.}$  depends on the solution  $X_0$  considered.

The differential Galois group of the order-four linear differential operator  $L_4$ , occurring in  $\tilde{\chi}^{(6)}$ , is in the symplectic group  $Sp(4, \mathbb{C})$ . We call  $X_0$  any of its formal solutions at  $x = 0$  and  $X_j$ , the  $j^{\text{th}}$  derivative up to  $j = 3$ .  $Y_0$  is another solution with its derivatives  $Y_j$ ,  $j = 1, \dots, 3$ . We define

$$w_{i,j} = X_i \cdot Y_j - X_j \cdot Y_i, \quad i = 0, \dots, 3, \quad j = 0, \dots, 3, \quad j > i. \quad (\text{D.3})$$

The first integral of the order-four differential operator  $L_4$  reads

$$Q(X_0, X_1, X_2, X_3, Y_0, Y_1, Y_2, Y_3) = \text{const.} \quad (\text{D.4})$$

where

$$\begin{aligned} Q = & \frac{x^7(1-16x)^{10} \cdot P_{36}}{A_{26}(L_4)A_4(\tilde{L}_3)^3} \cdot w_{0,1} + \frac{x^8 \cdot (1-16x)^{11}(1-8x) \cdot P_{34}}{A_{26}(L_4)A_4(\tilde{L}_3)^3} \cdot w_{0,2} \\ & + \frac{x^9 \cdot (1-16x)^{12}(1-4x)(1-8x) \cdot P_{28}}{A_{26}(L_4) \cdot A_4(\tilde{L}_3)^2} \cdot w_{0,3} \\ & + \frac{x^9 \cdot (1-16x)^{12}(1-8x) \cdot P_{29}}{A_{26}(L_4) \cdot A_4(\tilde{L}_3)^2} \cdot w_{1,2} \\ & + \frac{x^{10} \cdot (1-16x)^{13}(1-4x)(1-8x) \cdot P_{23}}{A_{26}(L_4) \cdot A_4(\tilde{L}_3)} \cdot w_{1,3} \\ & + \frac{x^{11}(1-16x)^{14}(1-4x)^2(1-8x) \cdot P_{17}}{A_{26}(L_4)} \cdot w_{2,3}. \end{aligned}$$

$P_j(x)$  are polynomials of degree  $j$ . The numerical value of  $\text{const.}$  depends on the solutions  $X_0$  and  $Y_0$  considered.

**Appendix E. Minimal order versus non minimal order:  $F_3 \cdot F_2$** 

For a given series  $S(x)$  solution of a Fuchsian operator of order  $q$ , one can consider the family of *Fuchsian* linear differential operators of order  $Q > q$  annihilating this series, the degree  $D$  of the polynomial coefficients being taken as small as possible [1]. The Fuchsian operator of minimal order annihilating this series is unique, and rightdivides all the operators of the previous family. Let us denote by  $N$  the minimum number of coefficients needed to find the linear ODE in this family within the constraint that the order  $Q$  and degree  $D$  are given. We found (as an experimental result) that  $N$ , the order  $Q$  and the degree  $D$  of the operators in this family are related by a *linear relation*, we called an "ODE formula" (see section 3.1 of [1]).

Assume we have a series  $S(x)$ , known modulo a prime, for which we have produced an ODE whose "ODE formula [1]" reads

$$N = 14Q + 5D - 14 = (Q + 1) \cdot (D + 1) - f, \quad (\text{E.1})$$

where  $Q$  is the order and  $D$  is the degree of  $P_j(x)$ ,  $j = 0, \dots, Q$  of the ODE written in the form  $P_Q(x) \cdot x^Q \cdot D_x^Q + \dots + P_0(x)$ , and  $f$  is the number of the independent non-minimal order ODE with  $Q$  and  $D$  such that  $(Q + 1)(D + 1) < \text{the available number of the series terms of } S(x)$  (see [1] and Appendix B of [2] for the details). From the "ODE formula [1]" we see that the minimal order ODE annihilating  $S(x)$  is of order 5, and we call it  $\mathcal{F}_5$ . Among the many non minimal order ODEs there is one which needs the lesser terms in  $S(x)$  to be produced. This particular *non minimal order* linear ODE is the "optimum ODE", and has, for this example, the order eight, i.e.  $(Q_0 = 8, D_0 = 23, f_0 = 3)$ . In the calculations we may use this order-eight ODE and continue to call it  $\mathcal{F}_5$ , but in this section we call it  $\mathcal{F}_5^{nm}$ . Obviously,  $\mathcal{F}_5$  is a right factor of  $\mathcal{F}_5^{nm}$ .

The local exponents at  $x = 0$  of this linear ODE are 0, 0, 1, 2, 3. These exponents are obtained whatever the order of the ODE is, in minimal order (i.e.  $\mathcal{F}_5$ ), or in non minimal order (i.e.  $\mathcal{F}_5^{nm}$ ). The three exponents corresponding to the three extra solutions of the order-eight linear ODE  $\mathcal{F}_5^{nm}$  appear as *non rational numbers*. Recall that we are dealing with globally nilpotent differential equations [5]. They are therefore necessarily Fuchsian and have *rational local exponents at all the singular points*.

The general (analytic at  $x = 0$ ) solution  $\tilde{S}(x)$  of  $\mathcal{F}_5$  (or  $\mathcal{F}_5^{nm}$ ) depends on two free coefficients, say  $\alpha$  and  $\beta$ . The series  $S(x)$  is a particular combination of  $\tilde{S}(x)$ . With  $Q = Q_0 = 8$ ,  $D = D_0 = 23$ , if there are some values of  $(\alpha = \alpha_0, \beta = \beta_0)$  in the range  $[1, p_r]$ , for which  $f$  is greater than  $f_0 = 3$ , the linear differential operator  $\mathcal{F}_5$  has a right factor. The solution  $\tilde{S}(x)$  with  $(\alpha = \alpha_0, \beta = \beta_0)$  gives an ODE, whose "ODE formula" reads

$$7Q + 2D + 2 = (Q + 1) \cdot (D + 1) - f, \quad (\text{E.2})$$

telling that, indeed, there is an order-two right factor occurring in the linear differential operator  $\mathcal{F}_5$ . We call this factor  $F_2$ , when obtained as minimal order ODE, and  $F_2^{nm}$  if it is obtained in non minimal order.

Here begin the details of our Remark 3. To obtain the factor (call it  $F_3$ ) at the left of  $F_2$  in  $\mathcal{F}_5$ , we may just use the "rightdivision" command of DEtools in Maple, or act by  $F_2$  on the series  $S(x)$  to obtain a series and look for the linear ODE annihilating it. But assume that this is cumbersome, or not doable. Either the "rightdivision" command is not feasible, or the series  $F_2(S(x))$  has no more enough coefficients terms

to encode the remaining left factor. This what happens in the case of  $L_{21} \cdot \tilde{L}_2$  when  $\tilde{L}_2$  is of minimal order.

Let us give some details on the factorization  $\mathcal{F}_5 = F_3 \cdot F_2$ . Assume we have obtained  $F_3$  from the series  $F_2^{nm}(S(x))$ . The rational solution of the symmetric square of  $F_3$  (in minimal order or in non minimal order) will appear as

$$\frac{P_{32}(x)}{(1-4x)^7(1+4x)^7(1-2x)(1+2x)}. \quad (\text{E.3})$$

If we use the minimal order  $F_2$  to obtain  $F_3$ . The rational solution of the symmetric square of  $F_3$  (in minimal order or in non minimal order) will appear as

$$\frac{P_{34}(x)}{(1-4x)^5(1+4x)^5}. \quad (\text{E.4})$$

To obtain the rational solution given in (11), one has to divide by the coefficient of the higher derivative of  $F_2$ . This is because, we have used  $F_2$ , in non monic form, to mimic the situation of the large orders linear differential operators for which the non monic form is more tractable in the computations.

As far as the occurrence of a rational solution to the symmetric square of a left factor is concerned, it is irrelevant whether the right factor is of minimal order or in non minimal order.

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